END
DATE
FILMED
9-8N
DTIC

# A SCATTERING FRAMEWORK FOR DECENTRALIZED ESTIMATION PROBLEMS[†]

Bernard C. Levy[*]

David A. Castanon[**]

George C. Verghese[**]

Alan S. Willsky[*]

1.0

4.5
2.8
2.5
3.2
2.2
3.6
4.0
2.0

1.1

1.8

1.25
1.4
1.6

MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS 1963 A

## ABSTRACT

In this paper we develop a comprehensive framework for the study of decentralized estimation problems. This approach imbeds a decentralized estimation problem into an equivalent scattering problem, and makes use of the superposition principle to relate local and centralized estimates. Some decentralized filtering and smoothing algorithms are obtained for a simple estimation structure consisting of a central processor and of two local processors. The case when the local processors exchange some information is considered, as well as the case when the local state-space models differ from the central one.

March 1981

LEVEL

ADA098117

# A SCATTERING FRAMEWORK FOR DECENTRALIZED ESTIMATION PROBLEMS[†]

Bernard C. Levy[*]

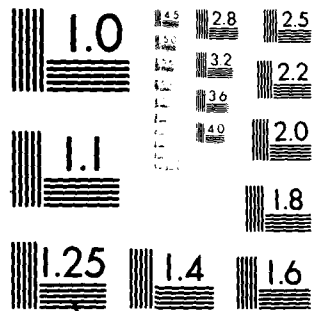David A. Castanon[*]

George C. Verghese[**]

Alan S. Willsky[*]

## I. Introduction

The study of complex, large-scale systems involves frequently the solution of decentralized estimation problems. This is the case for example when one considers some systems which are inherently decentralized, such as power systems [1], or when communications between subsystems are limited or unreliable, as in military command and control systems [2]. Also, in some applications, the volume of information to be processed is so large that a decentralized or parallel processing structure is required. Such structures have been proposed for example in image processing (if the strip Kalman filter discussed by Woods and Radewan [3]) or in order to estimate the location of moving sources in a distributed sensor network [4].

An additional incentive to consider decentralized estimation structures is the recent progress in VLSI (very large scale integration) technology which makes it now feasible to implement estimation algorithms in a parallel fashion [5]. However, in spite of these potential applications, it is only recently that decentralized estimation problems have been the object of some attention. This is due in part to the complexity of these problems: to describe entirely a decentralized estimation structure consisting of interconnected local processors one needs to specify for each processor a local model, a set of measurements and a local objective function (see Barta [6] for a general formulation of decentralized estimation problems). In addition, one needs to describe the topology and the capacity of the communications network connecting all

## ABSTRACT

In this paper we develop a comprehensive framework for the study of decentralized estimation problems. This approach imbeds a decentralized estimation problem into an equivalent scattering problem, and makes use of the superposition principle to relate local and centralized estimates. Some decentralized filtering and smoothing algorithms are obtained for a simple estimation structure consisting of a central processor and of two local processors. The case when the local processors exchange some information is considered, as well as the case when the local state-space models differ from the central one.

processors. These various elements aim at capturing the fact that in a decentralized estimation structure, each processor by itself has only a limited knowledge of the physical process under study, so that to overcome this limitation, the local processors must communicate. Also, even if all processors cooperate, their objectives may not be the same.

Another important feature of decentralized estimation structures is that they provide usually more than one way to solve a given problem. This explains why several approaches have been proposed to study decentralized estimation problems. Barta [6] has used a game-theoretic approach that allows for a large amount of flexibility in the choice of information flows between local processors. By contrast, Sanders et.al. [7] and Tacker and Sanders [8] have considered decentralized estimation filters with a fixed structure. Another approach, which was followed by Speyer [9], Chong [10] and Willsky et.al. [11] is based on decomposing a central estimation problem into smaller, local ones. This approach will be at the center of our discussion here.

The objective of this paper is to propose a more systematic framework for the study of decentralized estimation problems. To do so, we will use the scattering framework for linear estimation introduced by Kailath and his coworkers [12]-[15]. In this context, to every local or centralized linear estimation problem, we can associate an equivalent scattering problem which is specified by the knowledge of some scattering parameters and of a set of internal sources for the scattering medium. Then, if the scattering

## I. Introduction

The study of complex, large-scale systems involves frequently the solution of decentralized estimation problems. This is the case for example when one considers some systems which are inherently decentralized, such as power systems [1], or when communications between subsystems are limited or unreliable, as in military command and control systems [2]. Also, in some applications, the volume of information to be processed is so large that a decentralized or parallel processing structure is required. Such structures have been proposed for example in image processing (if the strip Kalman filter discussed by Woods and Radewan [3]) or in order to estimate the location of moving sources in a distributed sensor network [4].

An additional incentive to consider decentralized estimation structures is the recent progress in VLSI (very large scale integration) technology which makes it now feasible to implement estimation algorithms in a parallel fashion [5]. However, in spite of these potential applications, it is only recently that decentralized estimation problems have been the object of some attention. This is due in part to the complexity of these problems: to describe entirely a decentralized estimation structure consisting of interconnected local processors one needs to specify for each processor a local model, a set of measurements and a local objective function (see Barta [6] for a general formulation of decentralized estimation problems). In addition, one needs to describe the topology and the capacity of the communications network connecting all

parameters describing the various local or central estimation problems can be transformed so that they give rise to the same media, we can use the superposition principle for linear scattering media to relate the local and central estimates. This technique will be used here to obtain several decentralized estimation algorithms. To illustrate these results we will also consider several random fields problems of the type discussed in [11].

This paper is organized as follows. In Section II the scattering framework for linear estimation problems is introduced, and in Section III a brief review of the properties of scattering media is given. Then, in Section IV we obtain some decentralized smoothing and filtering algorithms for a simple decentralized structure consisting of a central processor and of two local processors which do not communicate between each other. When these local processors are allowed to communicate, the results of Section IV need to be modified as shown in Section V where we consider several problems, the smoothing update and the real-time smoothing problem, which appear in this context. The case when the state-space models available to the local processors differ from the global model is also considered in Section VI. Finally, in Section VII, we discuss some possible extensions of this work to networks where delays or noise can appear in the communication channels between processors.

These constitute only a few of the issues appearing in the context of decentralized estimation. In a companion paper, the scattering framework will also be used to obtain some decentralized algorithms for weakly coupled and singular perturbed systems.

processors. These various elements aim at capturing the fact that in a decentralized estimation structure, each processor by itself has only a limited knowledge of the physical process under study, so that to overcome this limitation, the local processors must communicate. Also, even if all processors cooperate, their objectives may not be the same.

Another important feature of decentralized estimation structures is that they provide usually more than one way to solve a given problem. This explains why several approaches have been proposed to study decentralized estimation problems. Barta [6] has used a game-theoretic approach that allows for a large amount of flexibility in the choice of information flows between local processors. By contrast, Sanders et.al. [7] and Tacker and Sanders [8] have considered decentralized estimation filters with a fixed structure. Another approach, which was followed by Speyer [9], Chong [10] and Willsky et.al. [11] is based on decomposing a central estimation problem into smaller, local ones. This approach will be at the center of our discussion here.

The objective of this paper is to propose a more systematic framework for the study of decentralized estimation problems. To do so, we will use the scattering framework for linear estimation introduced by Kailath and his coworkers [12]-[15]. In this context, to every local or centralized linear estimation problem, we can associate an equivalent scattering problem which is specified by the knowledge of some scattering parameters and of a set of internal sources for the scattering medium. Then, if the scattering

## II.  The scattering framework for filtering and smoothing problems

The scattering framework for the study of linear estimation problems has been discussed in great detail in [12]-[15].  We will therefore give here only a brief presentation of the main aspects of this theory.  We consider the state-space model

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad 0 \le t \le T \tag{2.1}$$

with observations

$$y(t) = Cx(t) + v(t) \tag{2.2}$$

where $u(t)$ and $v(t)$ are some white noise processes uncorrelated with the initial conditions $x(0)$, i.e.

$$E\left[\begin{pmatrix} u(t) \\ v(t) \end{pmatrix}\right] = 0, \qquad E\left[\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} x'(0)\right] = 0 \tag{2.3}$$

and

$$E\left[\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} (u'(s)v'(s))\right] = \begin{pmatrix} I_m & 0 \\ 0 & R \end{pmatrix} \delta(t-s) \tag{2.4}$$

with $R=R'>0$  (the estimation problem is assumed to be nonsingular). The a priori information on the initial conditions is given by

$$E[x(0)] = \hat{x}(0), \qquad E[\tilde{x}(0)\tilde{x}'(0)] = P(0) \tag{2.5}$$

where $\tilde{x}(0) = x(0) - \hat{x}(0)$.  In the following, it will be convenient to view this information as an additional observation on $x(0)$ given by

$$m = \hat{x}(0) = x(0) + w \tag{2.6}$$

parameters describing the various local or central estimation problems can be transformed so that they give rise to the same media, we can use the superposition principle for linear scattering media to relate the local and central estimates. This technique will be used here to obtain several decentralized estimation algorithms. To illustrate these results we will also consider several random fields problems of the type discussed in [11].

This paper is organized as follows. In Section II the scattering framework for linear estimation problems is introduced, and in Section III a brief review of the properties of scattering media is given. Then, in Section IV we obtain some decentralized smoothing and filtering algorithms for a simple decentralized structure consisting of a central processor and of two local processors which do not communicate between each other. When these local processors are allowed to communicate, the results of Section IV need to be modified as shown in Section V where we consider several problems, the smoothing update and the real-time smoothing problem, which appear in this context. The case when the state-space models available to the local processors differ from the global model is also considered in Section VI. Finally, in Section VII, we discuss some possible extensions of this work to networks where delays or noise can appear in the communication channels between processors.

These constitute only a few of the issues appearing in the context of decentralized estimation. In a companion paper, the scattering framework will also be used to obtain some decentralized algorithms for weakly coupled and singular perturbed systems.

where $E[w]=0$ and $E[ww']=p(0)$. Then, if we introduce the Hilbert spce

$$X = H(x(t), 0 \leq t \leq T), \qquad x_0 = H(x(0))$$

$$U = H(u(t), 0 \leq t \leq T), \qquad V = H(v(t), 0 \leq t \leq T)$$

$$Y = H(y(t), 0 \leq t \leq T)$$

the space $Y$ of observations is included in the space $G = x_0 \oplus U \oplus V = X \oplus V$ which specifies the total information contained in the system (2.1)-(2.2), i.e.

$$Y \subset G, \tag{2.7}$$

but in general this inclusion is strict, so that from the observations $Y$ one cannot recover completely $x(t)$ and $v(t)$ for $0 \leq t \leq T$.

However, as was observed by Weinert and Desai [16], the information lost by $Y$ can be entirely recovered by constructing the dual process $z(\cdot)$ defined by

$$\lambda(t) = -A'\lambda(t) + C'R^{-1}v(t), \qquad \lambda(T)=0$$

$$z(t) = B'\lambda(t) + u(t) \tag{2.8}$$

with the observations

$$n = P(0)\lambda(0) - w . \tag{2.9}$$

In this case, if

$$Z = H(n; z(t), 0 \leq t \leq T)$$

denotes the Hilbert space spanned by the dual process, one can show the following result.

## II. The scattering framework for filtering and smoothing problems

The scattering framework for the study of linear estimation problems has been discussed in great detail in [12]-[15]. We will therefore give here only a brief presentation of the main aspects of this theory. We consider the state-space model

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad 0 \le t \le T \tag{2.1}$$

with observations

$$y(t) = Cx(t) + v(t) \tag{2.2}$$

where $u(t)$ and $v(t)$ are some white noise processes uncorrelated with the initial conditions $x(0)$, i.e.

$$E\left[\begin{pmatrix} u(t) \\ v(t) \end{pmatrix}\right] = 0, \qquad E\left[\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} x'(0)\right] = 0 \tag{2.3}$$

and

$$E\left[\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} (u'(s)v'(s))\right] = \begin{pmatrix} I_m & 0 \\ 0 & R \end{pmatrix} \delta(t-s) \tag{2.4}$$

with $R=R'>0$ (the estimation problem is assumed to be nonsingular). The a priori information on the initial conditions is given by

$$E[x(0)] = \hat{x}(0), \qquad E[\tilde{x}(0)\tilde{x}'(0)] = P(0) \tag{2.5}$$

where $\tilde{x}(0) = x(0) - \hat{x}(0)$. In the following, it will be convenient to view this information as an additional observation on $x(0)$ given by

$$m = \hat{x}(0) = x(0) + w \tag{2.6}$$

**Lemma:** Z is the orthogonal complement of Y in G, i.e.

(i)     $Z \perp Y$ and (ii)  $Z \oplus Y = G$

The proof of (i) is obtained by direct verification (see also [16]). To prove (ii) one needs only to note that the system

$$\begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & -A' \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} + \begin{pmatrix} B & 0 \\ 0 & C'R^{-1} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{2.10}$$

with observations

$$\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} C & 0 \\ 0 & B' \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} + \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{2.11}$$

is invertible.  The inverse system is given by

$$\begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} A & -BB' \\ -C'R^{-1}C & -A' \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} + \begin{pmatrix} 0 & B \\ C'R^{-1} & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} \tag{2.12}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = -\begin{pmatrix} 0 & B' \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} + \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} \tag{2.13}$$

with the boundary conditions

$$\lambda(T) = 0 \qquad x(0) + P(0)\lambda(0) = m+n . \tag{2.14}$$

The solution of the two-point boundary value problem (2.12)-(2.14) enables us to reconstruct entirely  $G = X_0 \oplus U \oplus V$  from Y and Z so that (ii) is satisfied.

where $E[w]=0$ and $E[ww']=p(0)$. Then, if we introduce the Hilbert spce

$$X = H(x(t), \; 0 \leq t \leq T), \qquad x_0 = H(x(0))$$

$$U = H(u(t), \; 0 \leq t \leq T), \qquad V = H(v(t), \; 0 \leq t \leq T)$$

$$Y = H(y(t), \; 0 \leq t \leq T)$$

the space $Y$ of observations is included in the space $G = x_0 \oplus U \oplus V = X \oplus V$ which specifies the total information contained in the system (2.1)-(2.2), i.e.

$$Y \subset G, \tag{2.7}$$

but in general this inclusion is strict, so that from the observations $Y$ one cannot recover completely $x(t)$ and $v(t)$ for $0 \leq t \leq T$.

However, as was observed by Weinert and Desai [16], the information lost by $Y$ can be entirely recovered by constructing the __dual process__ $z(\cdot)$ defined by

$$\lambda(t) = -A'\lambda(t) + C'R^{-1}v(t), \qquad \lambda(T)=0$$
$$z(t) = B'\lambda(t) + u(t) \tag{2.8}$$

with the observations

$$n = P(0)\lambda(0) - w . \tag{2.9}$$

In this case, if

$$Z = H(n; z(t), \; 0 \leq t \leq T)$$

denotes the Hilbert space spanned by the dual process, one can show the following result.

This result can be used to obtain the scattering framework for the smoothing and filtering problems. By conditioning equations (2.12) and (2.14) with respect to $Y$, and by using the fact that $Y \perp Z$, one has

$$\begin{pmatrix} \dot{\hat{x}}(t|T) \\ \lambda(t|T) \end{pmatrix} = \begin{pmatrix} A & -BB' \\ -C'R^{-1}C & -A \end{pmatrix} \begin{pmatrix} \hat{x}(t|T) \\ \lambda(t|T) \end{pmatrix} + \begin{pmatrix} 0 \\ C'R^{-1}y(t) \end{pmatrix} \quad (2.15)$$

with $\lambda(T|T)=0$ and

$$\hat{x}(0|T) + P(0)\lambda(0|T) = m = \hat{x}(0) . \quad (2.16)$$

Here $x(t|T)$ and $\lambda(t|T)$ denote the linear least-squares smoothed estimates of $x(t)$ and $\lambda(t)$ given $m$ and $y(s)$, $0 \leq s \leq T$. Then, by discretizing (2.15), one gets

$$\begin{pmatrix} \hat{x}(t+\Delta|T) \\ \lambda(t|T) \end{pmatrix} = \underbrace{\begin{pmatrix} I+A\Delta & -BB'\Delta \\ C'R^{-1}C\Delta & I+A'\Delta \end{pmatrix}}_{S(t+\Delta,t)} \begin{pmatrix} \hat{x}(t|T) \\ \lambda(t+\Delta|T) \end{pmatrix} \quad (2.17)$$

$$+ \underbrace{\begin{pmatrix} 0 \\ -C'R^{-1}y(t)\Delta \end{pmatrix}}_{q(t)\Delta} + 0(\Delta^2),$$

and one can view (2.17) as obtained from a scattering medium with infinitesimal scattering matrix $S(t+\Delta,t)$, and interval sources $q(t)\Delta$ (see Figure 2.1).

**Lemma:** Z is the orthogonal complement of Y in G, i.e.

(i)    $Z \perp Y$ and (ii)  $Z \oplus Y = G$

The proof of (i) is obtained by direct verification (see also [16]).
To prove (ii) one needs only to note that the system

$$\begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & -A' \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} + \begin{pmatrix} B & 0 \\ 0 & C'R^{-1} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{2.10}$$

with observations

$$\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} C & 0 \\ 0 & B' \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} + \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{2.11}$$

is invertible.  The inverse system is given by

$$\begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} A & -BB' \\ -C'R^{-1}C & -A' \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} + \begin{pmatrix} 0 & B \\ C'R^{-1} & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} \tag{2.12}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = -\begin{pmatrix} 0 & B' \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} + \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} \tag{2.13}$$

with the boundary conditions

$$\lambda(T) = 0 \qquad x(0) + P(0)\lambda(0) = m+n . \tag{2.14}$$

The solution of the two-point boundary value problem (2.12)-(2.14)
enables us to reconstruct entirely  $G = X_0 \oplus U \oplus V$  from Y and Z
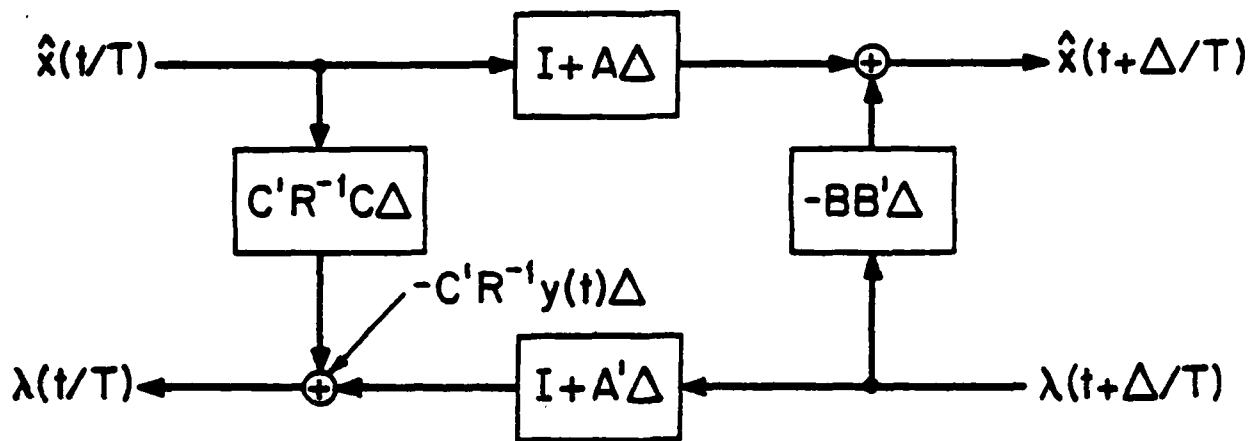so that (ii) is satisfied.

**Figure 2.1.** Scattering representation of the discretized smoothing equations.
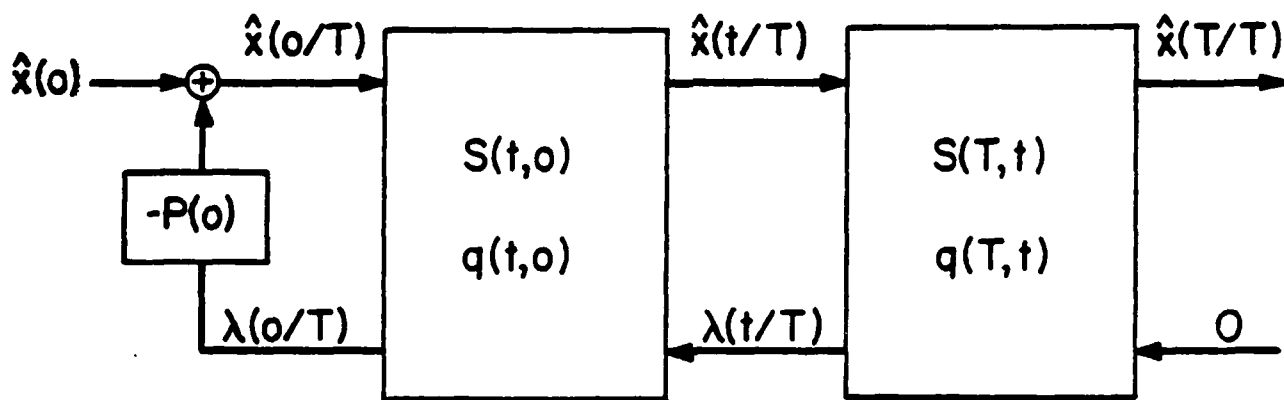


**Figure 2.2.** The scattering problem associated to the fixed-interval smoothing estimates.

This result can be used to obtain the scattering framework for the smoothing and filtering problems. By conditioning equations (2.12) and (2.14) with respect to  Y, and by using the fact that  $Y \perp Z$, one has

$$\begin{pmatrix} \dot{\hat{x}}(t|T) \\ \lambda(t|T) \end{pmatrix} = \begin{pmatrix} A & -BB' \\ -C'R^{-1}C & -A \end{pmatrix} \begin{pmatrix} \hat{x}(t|T) \\ \lambda(t|T) \end{pmatrix} + \begin{pmatrix} 0 \\ C'R^{-1}y(t) \end{pmatrix} \tag{2.15}$$

with $\lambda(T|T)=0$ and

$$\hat{x}(0|T) + P(0)\lambda(0|T) = m = \hat{x}(0) . \tag{2.16}$$

Here $x(t|T)$ and $\lambda(t|T)$ denote the linear least-squares smoothed estimates of $x(t)$ and $\lambda(t)$ given  m and $y(s)$, $0 \le s \le T$.   Then, by discretizing (2.15), one gets

$$\begin{pmatrix} \hat{x}(t+\Delta|T) \\ \lambda(t|T) \end{pmatrix} = \underbrace{\begin{pmatrix} I+A\Delta & -BB'\Delta \\ C'R^{-1}C\Delta & I+A'\Delta \end{pmatrix}}_{S(t+\Delta, t)} \begin{pmatrix} \hat{x}(t|T) \\ \lambda(t+\Delta|T) \end{pmatrix} \tag{2.17}$$

$$+ \underbrace{\begin{pmatrix} 0 \\ -C'R^{-1}y(t)\Delta \end{pmatrix}}_{q(t)\Delta} + O(\Delta^2),$$

and one can view (2.17) as obtained from a scattering medium with infinitesimal scattering matrix $S(t+\Delta,t)$, and interval sources $q(t)\Delta$ (see Figure 2.1).

By composing together these infinitesimal layers, we obtain an aggregate scattering medium which is described in Figure 2.2 (cf. Section III for a description of the rules of composition of scattering layers). The incoming waves for this medium are given by the boundary conditions (2.16), and the solution of the scattering problem requires the computation of the internal variables $(\hat{x}(t|T), \lambda(t|T))$. This can be done in a variety of ways which are discussed in detail in [14], [15]. Among existing methods, a method which is particularly simple is the <u>Mayne-Fraser two-filter formula</u> [17], [18]

$$P_S^{-1}(t)\hat{x}(t\ T) = P_F^{-1}(t)\hat{x}_F(t) + P_B^{-1}(t)\hat{x}_B(t) \tag{2.18}$$

where $P_S^{-1}(t) = P_F^{-1}(t) + P_B^{-1}(t)$ and where $(\hat{x}_F, P_F)$ and $(\hat{x}_B, P_B)$ denote respectively the forward and backward linear least-squares filtering estimates and error covariances of $x(t)$. They satisfy the equations

$$\dot{\hat{x}}_F(t) = (A - P_F C'R^{-1}C)\hat{x}_F(t) + P_F C'R^{-1}y(t) \tag{2.19}$$

$$\hat{x}_F(0) = \hat{x}(0)$$

and

$$-\dot{\hat{x}}_B(t) = (-A - P_B C'R^{-1}C)\hat{x}_B(t) + P_B C'R^{-1}y(t)$$

$$P_B^{-1}(T)\hat{x}_B(T) = 0 \tag{2.20}$$

with

$$\dot{P}_F = AP_F + P_F A' + BB' - P_F C'R^{-1}CP_F , \qquad P_F(0) = P(0)$$

$$\tag{2.21}$$

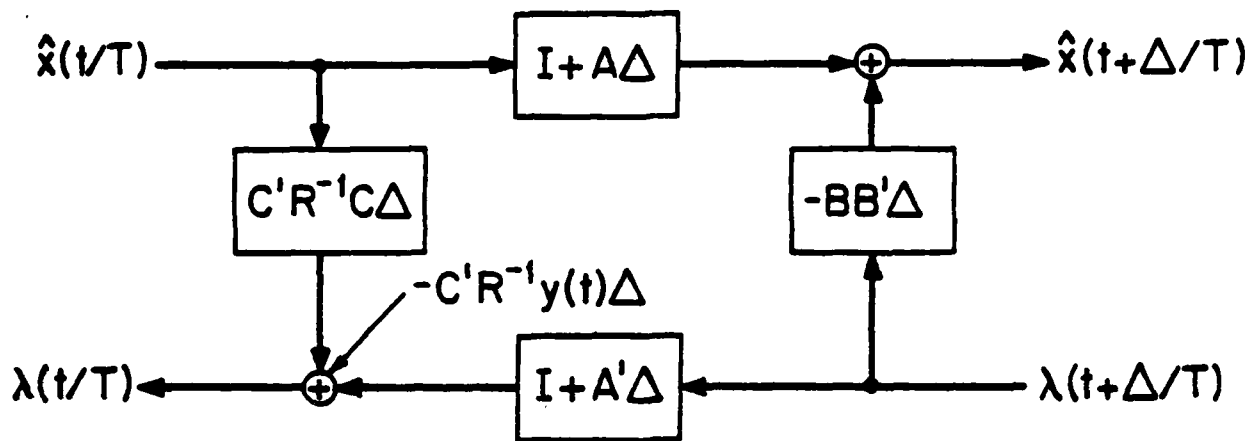$$-\dot{P}_B = -AP_B - P_B A' + BB' - P_B C'R^{-1}CP_B , \qquad P_B^{-1}(T) = 0 .$$

Figure 2.1. Scattering representation of the discretized smoothing equations.
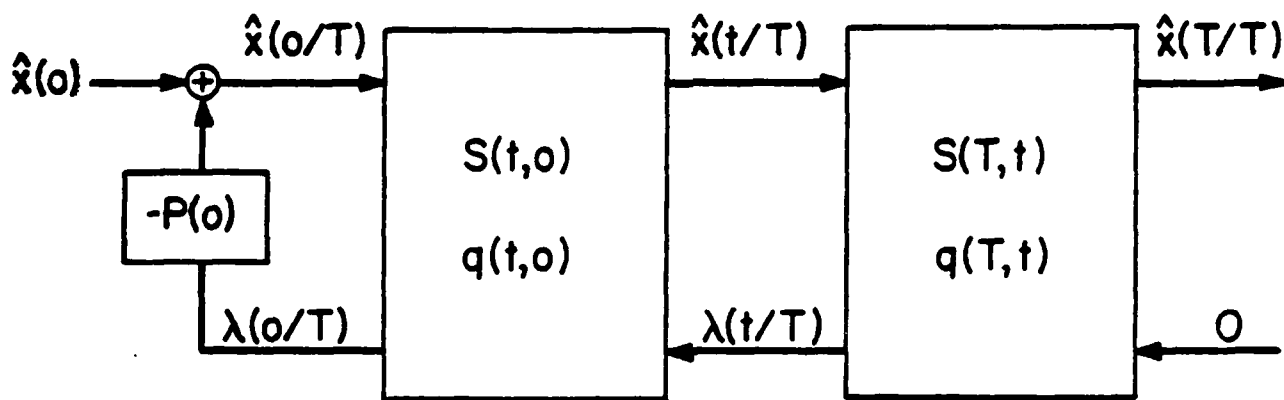


Figure 2.2. The scattering problem associated to the fixed-interval smoothing estimates.

A detailed discussion of this formula can be found in Wall, Willsky and Sandell [19].

This shows how to compute the smoothed estimate $\hat{x}(t|T)$. To compute the filtered estimate $\hat{x}(t|t) = \hat{x}_F(t)$, one needs only to use (2.19). By discretizing (2.19) and by dropping subscripts, one gets

$$\hat{x}(t+\Delta, t+\Delta) = (I+(A-PC'R^{-1}C)\Delta)\hat{x}(t|t) + PC'R^{-1}y(t)\Delta \qquad (2.22)$$

which can be viewed as obtained from a transmission medium with infinitesimal transmission matrix $T(t+\Delta,t) = I + (A-PC'R^{-1}C)\Delta$ and with internal sources $PC'R^{-1}y(t)\Delta$ (see Figure 2.3). These infinitesimal layers generate an aggregate transmission medium with initial $\hat{x}(0)$, and the solution of the transmission problem requires the computation of the internal variables $\hat{x}(t|t)$.

Instead of solving this problem, it is sometimes more convenient to consider the __information form__ of the Kalman filter, i.e.

$$\dot{d}(t) = (-A-P^{-1}BB')d(t) + C'R^{-1}y(t) \qquad (2.23)$$

where $d(t) \overset{\Delta}{=} P^{-1}(t)\hat{x}(t\ t)$ and $d(0) = P^{-1}(0)\hat{x}(0)$. The transmission problem associated to this form has the advantage that its infinitesimal sources are $C'R^{-1}y(t)\Delta$, so that they depend only on the original model instead of introducing $P(t)$ as well. This property will be exploited in Section IV to obtain some decentralized filtering algorithms.

By composing together these infinitesimal layers, we obtain an aggregate scattering medium which is described in Figure 2.2 (cf. Section III for a description of the rules of composition of scattering layers). The incoming waves for this medium are given by the boundary conditions (2.16), and the solution of the scattering problem requires the computation of the internal variables $(\hat{x}(t|T), \lambda(t|T))$. This can be done in a variety of ways which are discussed in detail in [14], [15]. Among existing methods, a method which is particularly simple is the <u>Mayne-Fraser two-filter formula</u> [17], [18]

$$P_S^{-1}(t)\hat{x}(t \mid T) = P_F^{-1}(t)\hat{x}_F(t) + P_B^{-1}(t)\hat{x}_B(t) \tag{2.18}$$

where $P_S^{-1}(t) = P_F^{-1}(t) + P_B^{-1}(t)$ and where $(\hat{x}_F, P_F)$ and $(\hat{x}_B, P_B)$ denote respectively the forward and backward linear least-squares filtering estimates and error covariances of $x(t)$. They satisfy the equations

$$\dot{\hat{x}}_F(t) = (A - P_F C'R^{-1}C)\hat{x}_F(t) + P_F C'R^{-1}y(t) \tag{2.19}$$

$$\hat{x}_F(0) = \hat{x}(0)$$

and

$$-\dot{\hat{x}}_B(t) = (-A - P_B C'R^{-1}C)\hat{x}_B(t) + P_B C'R^{-1}y(t)$$

$$P_B^{-1}(T)\hat{x}_B(T) = 0 \tag{2.20}$$

with

$$\dot{P}_F = AP_F + P_F A' + BB' - P_F C'R^{-1}CP_F , \qquad P_F(0) = P(0)$$

$$-\dot{P}_B = -AP_B - P_B A' + BB' - P_B C'R^{-1}CP_B , \qquad P_B^{-1}(T) = 0 . \tag{2.21}$$

$$\hat{x}(t/t) \longrightarrow \boxed{I + (A - PC'R^{-1}C)\Delta} \longrightarrow \oplus \longrightarrow \hat{x}(t+\Delta/t+\Delta)$$
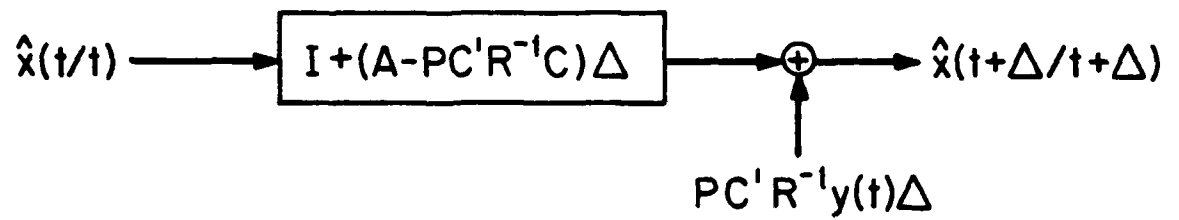
$$PC'R^{-1}y(t)\Delta$$

Figure 2.3.   The infinitesimal transmission layers associated to the filtering problem.

A detailed discussion of this formula can be found in Wall, Willsky and Sandell [19].

This shows how to compute the smoothed estimate $\hat{x}(t|T)$. To compute the filtered estimate $\hat{x}(t|t) = \hat{x}_F(t)$, one needs only to use (2.19). By discretizing (2.19) and by dropping subscripts, one gets

$$\hat{x}(t+\Delta, t+\Delta) = (I+(A-PC'R^{-1}C)\Delta)\hat{x}(t|t) + PC'R^{-1}y(t)\Delta \qquad (2.22)$$

which can be viewed as obtained from a transmission medium with infinitesimal transmission matrix $T(t+\Delta,t) \approx I + (A-PC'R^{-1}C)\Delta$ and with internal sources $PC'R^{-1}y(t)\Delta$ (see Figure 2.3). These infinitesimal layers generate an aggregate transmission medium with initial $\hat{x}(0)$, and the solution of the transmission problem requires the computation of the internal variables $\hat{x}(t|t)$.

Instead of solving this problem, it is sometimes more convenient to consider the _information form_ of the Kalman filter, i.e.

$$\dot{d}(t) = (-A-P^{-1}BB')d(t) + C'R^{-1}y(t) \qquad (2.23)$$

where $d(t) \overset{\Delta}{=} P^{-1}(t)\hat{x}(t|t)$ and $d(0) = P^{-1}(0)\hat{x}(0)$. The transmission problem associated to this form has the advantage that its infinitesimal sources are $C'R^{-1}y(t)\Delta$, so that they depend only on the original model instead of introducing $P(t)$ as well. This property will be exploited in Section IV to obtain some decentralized filtering algorithms.

## III. Some facts from scattering theory

The algebra of scattering matrices was first introduced by Redheffer [20], [21], and a discussion of the main aspects of this theory can be found in [12]-[15]. Only a few basic facts from this theory will be required here.

A scattering medium is specified by a scattering matrix

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad (3.1)$$

and by some internal sources

$$Q = \begin{pmatrix} q_+ \\ q_- \end{pmatrix} \qquad (3.2)$$

(see Figure 3.1). The parameters a and b (which can be operators, or matrices) are called the transmission coefficients of the medium. They describe the portion of a wave $l_-$(resp. $r_-$) travelling from left to right (resp. from right to left) which is transmitted by the medium. Similarly, the parameters c and b are called the reflection coefficients and describe the parts of the waves $l_-$ and $r_-$ which are reflected. Then, the outgoing waves are given in function of the incoming waves by

$$\begin{pmatrix} r_+ \\ l_+ \end{pmatrix} = S \begin{pmatrix} l_- \\ r_- \end{pmatrix} + \begin{pmatrix} q_+ \\ q_- \end{pmatrix} \qquad (3.3)$$

It is assumed here that the scattering medium is linear.

$$\hat{x}(t/t) \longrightarrow \boxed{I + (A - PC'R^{-1}C)\Delta} \longrightarrow \bigoplus \longrightarrow \hat{x}(t+\Delta/t+\Delta)$$
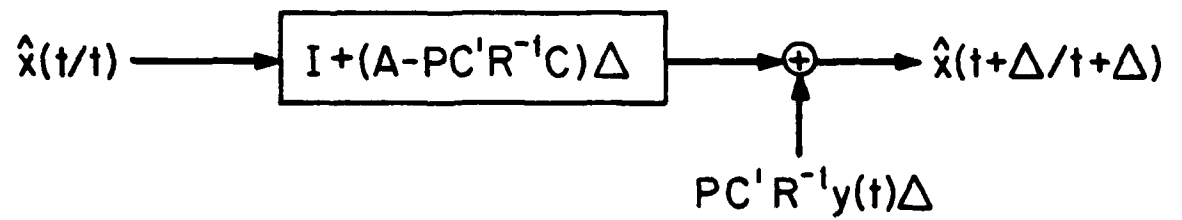
$$PC'R^{-1}y(t)\Delta$$

Figure 2.3.    The infinitesimal transmission layers associated
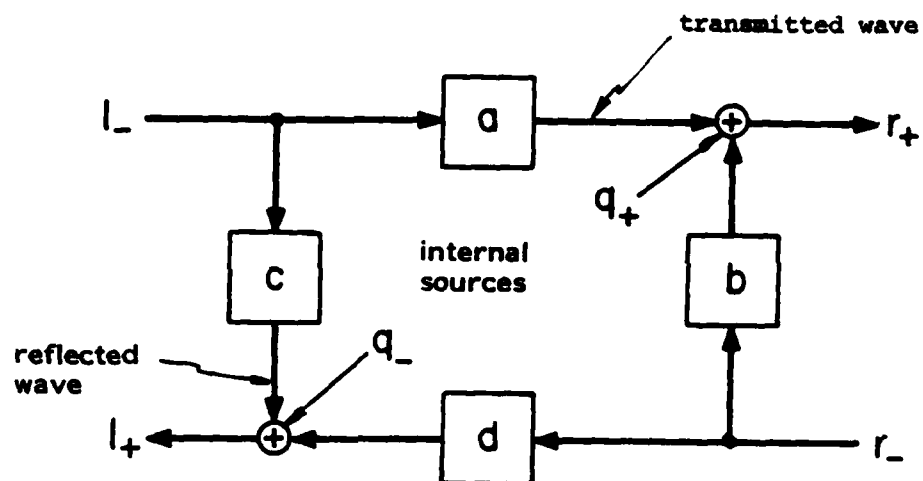to the filtering problem.

**Figure 3.1.** Description of a scattering medium.
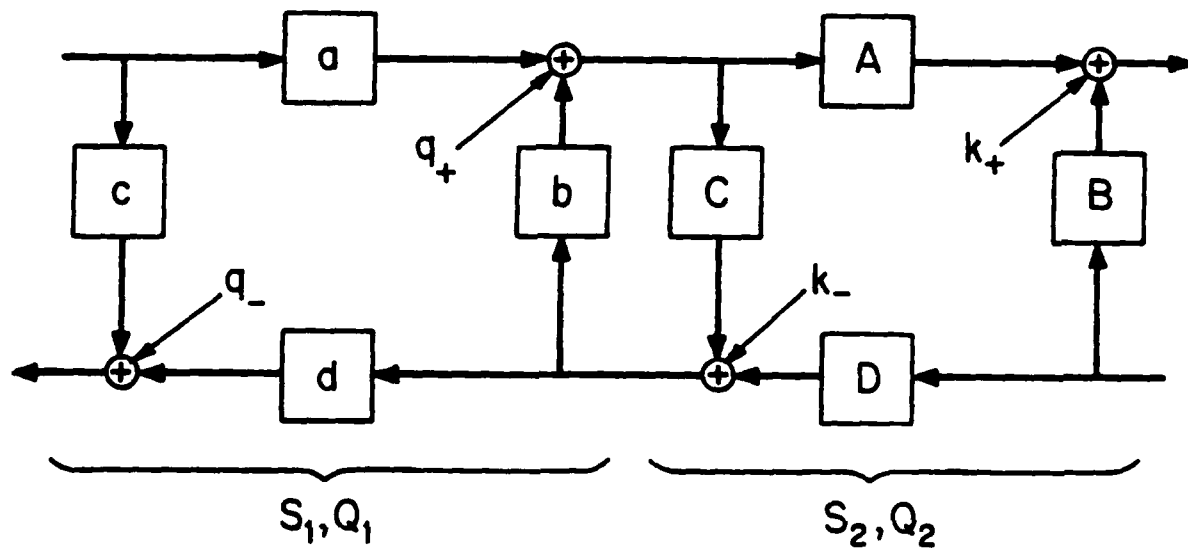


**Figure 3.2.** Composition of two scattering layers.

## III.  Some facts from scattering theory

The algebra of scattering matrices was first introduced by Redheffer [20], [21], and a discussion of the main aspects of this theory can be found in [12]-[15].  Only a few basic facts from this theory will be required here.

A scattering medium is specified by a scattering matrix

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3.1}$$

and by some internal sources

$$Q = \begin{pmatrix} q_+ \\ q_- \end{pmatrix} \tag{3.2}$$

(see Figure 3.1).  The parameters  a  and b (which can be operators, or matrices) are called the transmission coefficients of the medium.  They describe the portion of a wave $l_-$(resp. $r_-$) travelling from left to right (resp. from right to left) which is transmitted by the medium. Similarly, the parameters  c and  b are called the reflection coefficients and describe the parts of the waves $l_-$ and $r_-$  which are reflected.  Then, the outgoing waves are given in function of the incoming waves by

$$\begin{pmatrix} r_+ \\ l_+ \end{pmatrix} = S \begin{pmatrix} l_- \\ r_- \end{pmatrix} + \begin{pmatrix} q_+ \\ q_- \end{pmatrix} \tag{3.3}$$

It is assumed here that the scattering medium is linear.

To compose two scattering layers, one needs to introduce a special product different from the ordinary matrix product which is called the star product (see [20],[21]). For the scattering layers described in Figure 3.2, one gets

$$
S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} * \begin{pmatrix} a & b \\ c & d \end{pmatrix}
$$

$$
= \begin{pmatrix} A(I-bC)^{-1}a & B+Ab(I-Cb)^{-1}D \\ c+dC(I-bC)^{-1}a & d(I-Cb)^{-1}D \end{pmatrix} \tag{3.4}
$$

where the transmission and reflection coefficients of S can be obtain by using standard flow graph rules [22]. To compose internal sources, one needs also to use a special sum called the assembly sum. For the internal sources

$$
Q_1 = \begin{pmatrix} q_+ \\ q_- \end{pmatrix} \quad \text{and} \quad Q_2 = \begin{pmatrix} k_+ \\ k_- \end{pmatrix} \tag{3.5}
$$

of Figure 3.2, it is denoted as $Q=Q_1 \circ Q_2$ where

$$
Q = \begin{pmatrix} Q_+ \\ Q_- \end{pmatrix} = \begin{pmatrix} k_+ \\ q_- \end{pmatrix} + \begin{pmatrix} A(I-bC)^{-1}(q_++bk_-) \\ d(I-Cb)^{-1}(k_-Cq_+) \end{pmatrix} \tag{3.6}
$$

**Figure 3.1.** Description of a scattering medium.
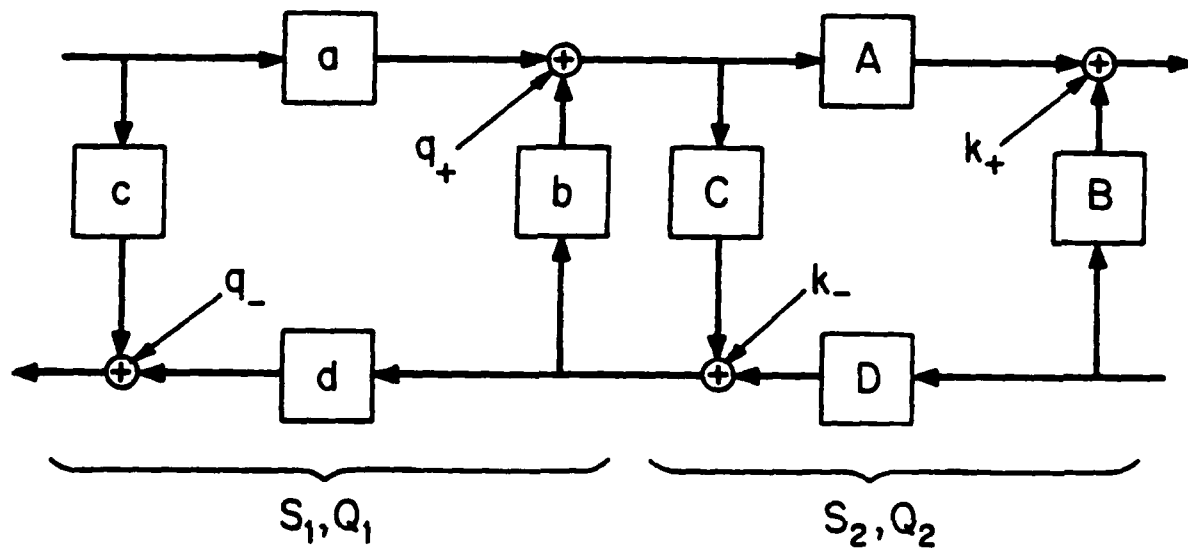


**Figure 3.2.** Composition of two scattering layers.

**Example:** If we consider the scattering medium generated by the smoothing problem of Section II, i.e.

$$S_1 = S(t,0) = \begin{pmatrix} \phi(t,0) & -P(t,0) \\ w(t,0) & \Psi(t,0) \end{pmatrix} \tag{3.7}$$

$$Q_1 = q(t,0) = \begin{pmatrix} q_+(t,0) \\ q_-(t,0) \end{pmatrix} \tag{3.8}$$

the previous composition rules can be used to obtain a set of differential equations for the entries of $S(t,0)$ and $q(t,0)$. Thus, if

$$S_2 = S(t+\Delta,t) = \begin{pmatrix} I+A\Delta & -BB'\Delta \\ C'R^{-1}C\Delta & I+A'\Delta \end{pmatrix} \tag{3.9}$$

and

$$Q_2 = q(t)\Delta = \begin{pmatrix} 0 \\ -C'R^{-1}y(t)\Delta \end{pmatrix} \tag{3.10}$$

one has

$$S_2 * S_1 = S(t+\Delta,0)$$

$$= \begin{pmatrix} \phi+(A-KC)\phi\Delta & P+(AP+PA'+BB'-PC'R^{-1}CP)\Delta \\ W+\Psi C'R^{-1}C\phi\Delta & \Psi+\Psi(A-KC)'\Delta \end{pmatrix} + 0(\Delta^2) \tag{3.11}$$

where $K(t) = P(t,0)C'R^{-1}$ and where the time arguments have been omitted. This shows that

To compose two scattering layers, one needs to introduce a special product different from the ordinary matrix product which is called the **star product** (see [20],[21]). For the scattering layers described in Figure 3.2, one gets

$$
S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} * \begin{pmatrix} a & b \\ c & d \end{pmatrix}
$$

$$
= \begin{pmatrix} A(I-bC)^{-1}a & B+Ab(I-Cb)^{-1}D \\ c+dC(I-bC)^{-1}a & d(I-Cb)^{-1}D \end{pmatrix} \tag{3.4}
$$

where the transmission and reflection coefficients of S can be obtain by using standard flow graph rules [22]. To compose internal sources, one needs also to use a special sum called the **assembly sum**. For the internal sources

$$
Q_1 = \begin{pmatrix} q_+ \\ q_- \end{pmatrix} \quad \text{and} \quad Q_2 = \begin{pmatrix} k_+ \\ k_- \end{pmatrix} \tag{3.5}
$$

of Figure 3.2, it is denoted as $Q=Q_1 \circ Q_2$ where

$$
Q = \begin{pmatrix} Q_+ \\ Q_- \end{pmatrix} = \begin{pmatrix} k_+ \\ q_- \end{pmatrix} + \begin{pmatrix} A(I-bC)^{-1}(q_+ +bk_-) \\ d(I-Cb)^{-1}(k_- Cq_+) \end{pmatrix} \tag{3.6}
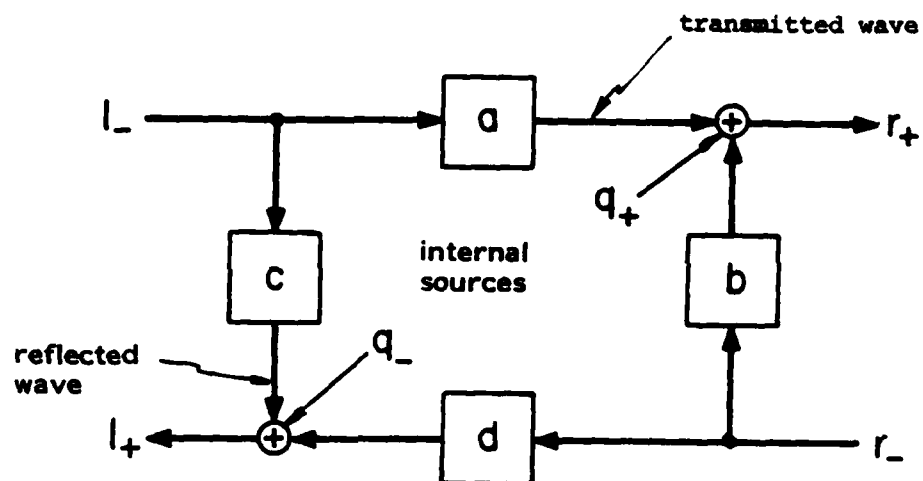$$

$$\dot{\phi}(t,0) = (A-KC)\phi \qquad\qquad , \quad \phi(0,0)=I$$

$$\dot{P}(t,0) = AP+PA'+BB'-PC'R^{-1}CP \qquad , \quad P(0,0)=0$$

$$\dot{W}(t,0) = \Psi C'R^{-1}C\phi \qquad\qquad , \quad W(0,0)=0 \qquad\qquad (3.12)$$

$$\dot{\Psi}(t,0) = \Psi(A-KC)' \qquad\qquad , \quad \Psi(0,0)=I$$

so that $P(t,0)$ is the solution of the Riccati equation associated to the filtering problem with zero initial conditions, $\phi(t,0) = \Psi'(t,0)$ is the transition matrix of the corresponding Kalman filter and

$$W(t,0) = \int_0^t \phi'(s,0)C'R^{-1}C\phi(s,0)ds \qquad\qquad (3.13)$$

is the associated observability matrix. By composing sources, one gets also

$$Q_1 \circ Q_2 = q(t+\Delta,0)$$

$$= \begin{pmatrix} q_+(t,0)+((A-KC)q_+(t,0)+Ky(t))\Delta \\ q_-(t,0)-\Psi C'R^{-1}(y(t)-Cq_+(t,0))\Delta \end{pmatrix} \qquad\qquad (3.14)$$

so that

$$\dot{q}_+(t,0) = (A-KC)q_+ + Ky \qquad , \quad q_+(0,0)=0$$

$$\dot{q}_-(t,0) = -\phi'C'R^{-1}(y-Cq_+) \qquad , \quad q_-(0,0)=0. \qquad\qquad (3.15)$$

This shows that

$$q_+(t,0) = \hat{x}_0(t|t) \qquad\qquad (3.16)$$

**Example:** If we consider the scattering medium generated by the smoothing problem of Section II, i.e.

$$S_1 = S(t,0) = \begin{pmatrix} \phi(t,0) & -P(t,0) \\ w(t,0) & \Psi(t,0) \end{pmatrix} \tag{3.7}$$

$$Q_1 = q(t,0) = \begin{pmatrix} q_+(t,0) \\ q_-(t,0) \end{pmatrix} \tag{3.8}$$

the previous composition rules can be used to obtain a set of differential equations for the entries of $S(t,0)$ and $q(t,0)$. Thus, if

$$S_2 = S(t+\Delta,t) = \begin{pmatrix} I+A\Delta & -BB'\Delta \\ C'R^{-1}C\Delta & I+A'\Delta \end{pmatrix} \tag{3.9}$$

and

$$Q_2 = q(t)\Delta = \begin{pmatrix} 0 \\ -C'R^{-1}y(t)\Delta \end{pmatrix} \tag{3.10}$$

one has

$$S_2 * S_1 = S(t+\Delta,0)$$

$$= \begin{pmatrix} \phi+(A-KC)\phi\Delta & P+(AP+PA'+BB'-PC'R^{-1}CP)\Delta \\ w+\Psi C'R^{-1}C\phi\Delta & \Psi+\Psi(A-KC)'\Delta \end{pmatrix} + O(\Delta^2) \tag{3.11}$$

where $K(t) = P(t,0)C'R^{-1}$ and where the time arguments have been omitted. This shows that

where $\hat{x}_0(t|t)$ denotes the Kalman filtering estimate obtained for zero initial conditions ($\hat{x}(0)=0$ and $P(0)=0$) and

$$q_-(t,0) = -\int_0^t \phi'(s,0)C'R^{-1}v_0(s)ds \qquad (3.17)$$

where $v_0(t) = y(t) - C\hat{x}_0(t|t)$ is the innovations process for zero initial conditions.

One drawback of the star product for composing scattering layers is that this product is _nonlinear_. This leads for example to some nonlinear differential equations such as (3.12) for the entries of $S(t,0)$. To linearize these equations, Redheffer noted in [20] that one needs only to transform the scattering medium of Figure 3.1 into a transmitting one. This can be done by expressing the waves $(r_+, r_-)$ at the right of the scattering layer in function of the waves $(l_-, l_+)$. By using flow-graph inversion rules as shown in Figure 3.3, one obtains

$$\begin{pmatrix} r_+ \\ r_- \end{pmatrix} = \underbrace{\begin{pmatrix} a-bd^{-1}c & bd^{-1} \\ -d^{-1}c & d^{-1} \end{pmatrix}}_{T} \begin{pmatrix} l_- \\ l_+ \end{pmatrix} + \underbrace{\begin{pmatrix} q_+ - bd^{-1}q_- \\ -d^{-1}q_- \end{pmatrix}}_{R} \qquad (3.18)$$

where T is the transmission matrix of the medium and where R describes the transmission sources. Then, the rules of composition of transmission layers are the same as those of ordinary matrix multiplication so that if a medium (T,R) is obtained by composing two layers

where $\hat{x}_0(t|t)$ denotes the Kalman filtering estimate obtained for zero initial conditions ($\hat{x}(0)=0$ and $P(0)=0$) and

$$q_-(t,0) = - \int_0^t \phi'(s,0)C'R^{-1}v_0(s)ds \qquad (3.17)$$

where $v_0(t) = y(t) - C\hat{x}_0(t|t)$ is the innovations process for zero initial conditions.

One drawback of the star product for composing scattering layers is that this product is _nonlinear_. This leads for example to some nonlinear differential equations such as (3.12) for the entries of $S(t,0)$. To linearize these equations, Redheffer noted in [20] that one needs only to transform the scattering medium of Figure 3.1 into a transmitting one. This can be done by expressing the waves $(r_+,r_-)$ at the right of the scattering layer in function of the waves $(l_-,l_+)$. By using flow-graph inversion rules as shown in Figure 3.3, one obtains

$$\begin{pmatrix} r_+ \\ r_- \end{pmatrix} = \underbrace{\begin{pmatrix} a-bd^{-1}c & bd^{-1} \\ -d^{-1}c & d^{-1} \end{pmatrix}}_{T} \begin{pmatrix} l_- \\ l_+ \end{pmatrix} + \underbrace{\begin{pmatrix} q_+ - bd^{-1}q_- \\ -d^{-1}q_- \end{pmatrix}}_{R} \qquad (3.18)$$

where T is the transmission matrix of the medium and where R describes the transmission sources. Then, the rules of composition of transmission layers are the same as those of ordinary matrix multiplication so that if a medium (T,R) is obtained by composing two layers

**Figure 3.3.** Transformation of a scattering medium into a transmitting one.
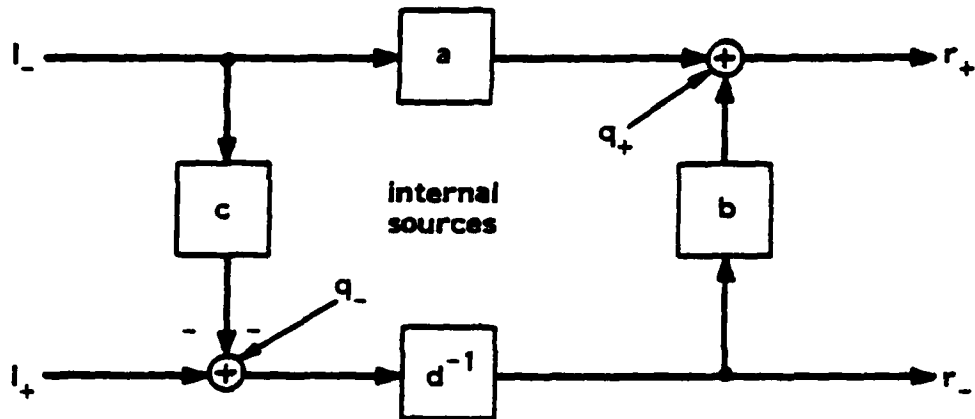
**Figure 3.3.** Transformation of a scattering medium into a transmitting one.

$(T_i, R_i)$, $i=1,2$ one has

$$T = T_2 T_1 , \qquad R = R_2 + T_2 R_1 . \qquad (3.19)$$

We note however that since scattering problems involve the specification of incoming waves at the boundaries of the medium, if we transform a scattering problem into a transmission one, one gets a two-point boundary value problem. Such problems have been studied by Reid [23], and more recently by Sidhu and Desai [24] in the context of linear estimation.

In the following, in order to solve scattering problems, we will make repeated use of the superposition principle. This principle applies only to linear scattering media. Let S be the scattering matrix of a linear medium, and let $(r_+^i, 1_+^i)$ be the outgoing waves of the medium associated to the incoming waves $(1_-^i, r_-^i)$ and internal sources

$$Q_i = \begin{pmatrix} q_+^i \\ q_-^i \end{pmatrix}$$

for $i=1,2$. Then, the superposition principle states that the outgoing waves associated to the incoming waves $(1_-, r_-) = (1_-^1 + 1_-^2, r_-^1 + r_-^2)$ and the sources $Q = Q_1 + Q_2$ are given by

$$(r_+, 1_+) = (r_+^1 + r_+^2, 1_+^1 + 1_+^2) . \qquad (3.20)$$

$(T_i, R_i)$, $i=1,2$  one has

$$T = T_2 T_1 \ , \qquad R = R_2 + T_2 R_1 \ . \qquad\qquad (3.19)$$

We note however that since scattering problems involve the specification of incoming waves at the boundaries of the medium, if we transform a scattering problem into a transmission one, one gets a two-point boundary value problem.  Such problems have been studied by Reid [23], and more recently by Sidhu and Desai [24] in the context of linear estimation.

In the following, in order to solve scattering problems, we will make repeated use of the superposition principle.  This principle applies only to linear scattering media.  Let S be the scattering matrix of a linear medium, and let  $(r_+^i, \ l_+^i)$  be the outgoing waves of the medium associated to the incoming waves $(l_-^i, r_-^i)$  and internal sources

$$Q_i = \begin{pmatrix} q_+^i \\[2mm] q_-^i \end{pmatrix}$$

for $i=1,2$.  Then, the superposition principle states that the outgoing waves associated to the incoming waves $(l_-, r_-) = (l_-^1 + l_-^2, r_-^1 + r_-^2)$ and the sources $Q = Q_1 + Q_2$  are given by

$$(r_+, l_+) = (r_+^1 + r_+^2, \ l_+^1 + l_+^2) \ . \qquad\qquad (3.20)$$

## IV. The decentralized smoothing and filtering problems

A simple decentralized estimation problem can be described as follows: let $S_i$ $i=1,2$ be two stations which observe independently a process

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad 0 \leq t \leq T . \qquad (4.1)$$

The measurements obtained by $S_i$ $i=1,2$ are denoted as

$$y_i(t) = C_i x(t) + v_i(t) \qquad (4.2)$$

where $v_1$ and $v_2$ are some independent white Gaussian noises such that

$$E\left[\begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} (v_1'(s)v_2'(s))\right] = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix} \delta(t-s) . \qquad (4.3)$$

Then, if we assume that $S_1$ and $S_2$ are linked to a central processor P, and if we want to compute the smoothed or filtered estimate of $x(t)$ given $y_1(\cdot)$ and $y_2(\cdot)$, two strategies are possible:

(i)    $S_1$ and $S_2$ send their measurements to the central processor which computes $\hat{x}$, the estimate of x given $y_1(\cdot)$ and $y_2(\cdot)$.

(ii)   $S_i$ $i=1,2$ processes its own information locally and computes $\hat{x}^{(i)}$ (the estimate of x given $y_i(\cdot)$). Then $\hat{x}^{(i)}$ $i=1,2$, is sent to the central processor which in turn seeks to combine $\hat{x}^{(1)}$ and $\hat{x}^{(2)}$ to obtain the centralized estimate $\hat{x}$.

The first of these strategies is centralized and any of the existing smoothing and filtering algorithms can be used to compute $\hat{x}$. The second one is decentralized and will be the object of our discussion here.

## IV.  The decentralized smoothing and filtering problems

A simple decentralized estimation problem can be described as follows:  let  $S_i$  i=1,2  be two stations which observe independently a process

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad 0 \leq t \leq T .  \qquad (4.1)$$

The measurements obtained by  $S_i$  i=1,2  are denoted as

$$y_i(t) = C_i x(t) + v_i(t)  \qquad (4.2)$$

where  $v_1$  and  $v_2$  are some independent white Gaussian noises such that

$$E\left[\begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} (v_1'(s) v_2'(s)) \right] = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix} \delta(t-s) .  \qquad (4.3)$$

Then, if we assume that  $S_1$  and  $S_2$  are linked to a central processor P, and if we want to compute the smoothed or filtered estimate of  x(t)  given  $y_1(\cdot)$  and  $y_2(\cdot)$ , two strategies are possible:

(i)     $S_1$  and  $S_2$  send their measurements to the central processor which computes  $\hat{x}$ , the estimate of x given  $y_1(\cdot)$  and  $y_2(\cdot)$ .

(ii)    $S_i$  i=1,2  processes its own information locally and computes  $\hat{x}^{(i)}$  (the estimate of x given  $y_i(\cdot)$ ).  Then  $\hat{x}^{(i)}$  i=1,2, is sent to the central processor which in turn seeks to combine  $\hat{x}^{(1)}$  and  $\hat{x}^{(2)}$  to obtain the centralized estimate  $\hat{x}$ .

The first of these strategies is centralized and any of the existing smoothing and filtering algorithms can be used to compute  $\hat{x}$ .  The second one is decentralized and will be the object of our discussion here.

## A. Decentralized smoothing

In the following, we will write as $Y_i = H(m; y_i(s), 0 \leq s \leq T)$ the Hilbert space spanned by the observations available to $S_i$ and we will denote by

$$\hat{x}(t|T) = E[x t)|Y_1, Y_2] \tag{4.4}$$

and

$$\hat{x}^{(i)}(t|T) = E[x(t)|Y_i] \quad i=1,2 \tag{4.5}$$

the centralized and local smoothed estimates of $x(t)$.

Then, by using the scattering framework of Section II, the estimates $\hat{x}(t|T)$ and $\hat{x}^{(i)}(t|T)$ can be obtain by composing some global and local scattering layers such as those depicted in Figure 4.1a and 4.1b. One important feature of these elementary scattering layers is that they correspond to nearly identical media. The only difference is that for the global medium, the reflection matrix associated to the left incoming wave $\hat{x}(t|T)$ is $(C_1' R_1^{-1} C_1 + C_2' R_2^{-1} C_2)\Delta$ and that for the local medium, the reflection matrix for $\hat{x}^{(i)}(t|T)$ is $C_1' R_1^{-1} C_1 \Delta$.

This means that we cannot apply directly the superposition principle to these scattering layers. However, a very simple transformation can be used to make the local medium identical to the global one: one needs only to replace the local reflection matrix for $\hat{x}^{(1)}(t|T)$ by $(C_1' R_1^{-1} C_1 + C_2' R_2^{-1} C_2)\Delta$ and compensate for the additional reflected wave $C_2' R_2^{-1} C_2 \hat{x}^{(1)}(t|T)\Delta$ by a fictitious source equal to $-C_2' R_2^{-1} C_2 \hat{x}^{(1)}(t|T)\Delta$ as shown in Figure 4.1c.

## A. Decentralized smoothing

In the following, we will write as $Y_i = H(m; y_i(s), 0 \leq s \leq T)$ the Hilbert space spanned by the observations available to $S_i$ and we will denote by

$$\hat{x}(t|T) = E[x\ t)|Y_1, Y_2] \tag{4.4}$$

and

$$\hat{x}^{(i)}(t|T) = E[x(t)|Y_i]\ i=1,2 \tag{4.5}$$

the centralized and local smoothed estimates of $x(t)$.

Then, by using the scattering framework of Section II, the estimates $\hat{x}(t|T)$ and $\hat{x}^{(i)}(t|T)$ can be obtain by composing some global and local scattering layers such as those depicted in Figure 4.1a and 4.1b. One important feature of these elementary scattering layers is that they correspond to nearly identical media. The only difference is that for the global medium, the reflection matrix associated to the left incoming wave $\hat{x}(t|T)$ is $(C_1'R_1^{-1}C_1 + C_2'R_2^{-1}C_2)\Delta$ and that for the local medium, the reflection matrix for $\hat{x}^{(i)}(t|T)$ is $C_1'R_1^{-1}C_1\Delta$.

This means that we cannot apply directly the superposition principle to these scattering layers. However, a very simple transformation can be used to make the local medium identical to the global one: one needs only to replace the local reflection matrix for $\hat{x}^{(1)}(t|T)$ by $(C_1'R_1^{-1}C_1 + C_2'R_2^{-1}C_2)\Delta$ and compensate for the additional reflected wave $C_2'R_2^{-1}C_2\hat{x}^{(1)}(t|T)\Delta$ by a fictitious source equal to $-C_2'R_2^{-1}C_2\hat{x}^{(1)}(t|T)\Delta$ as shown in Figure 4.1c.
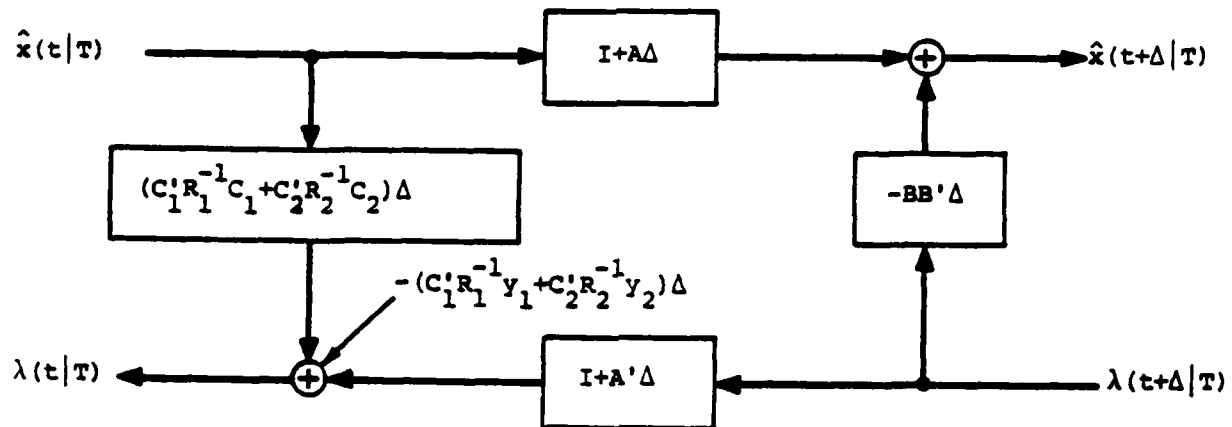
Figure 4.1a.   Global scattering layers.
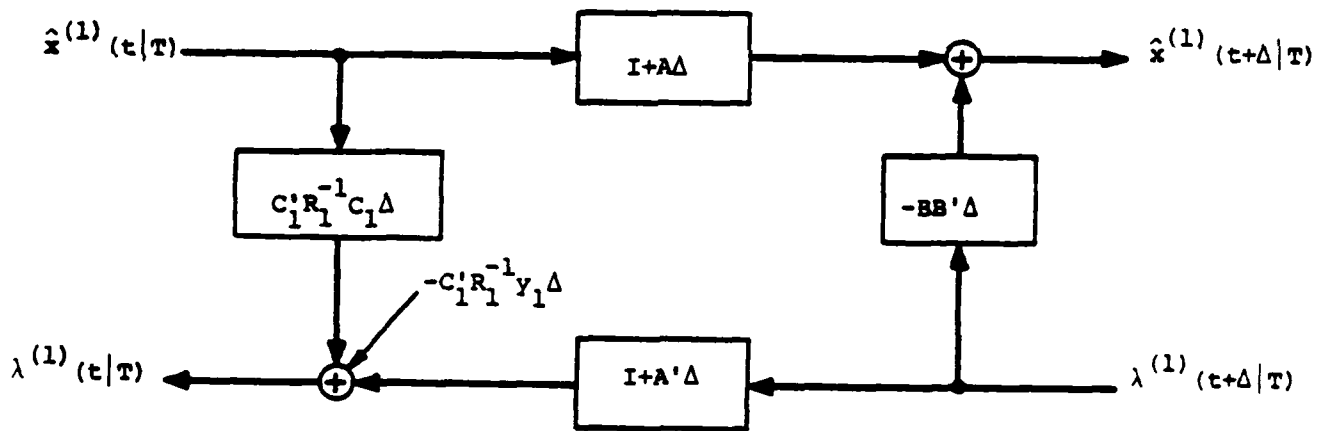


Figure 4.1b.   Local scattering layers for i=1.
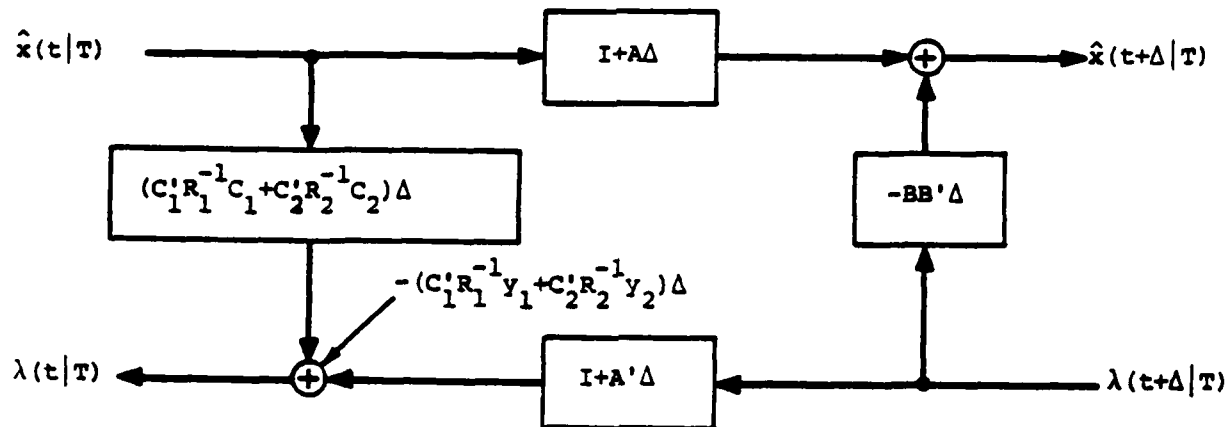
Figure 4.1a.    Global scattering layers.
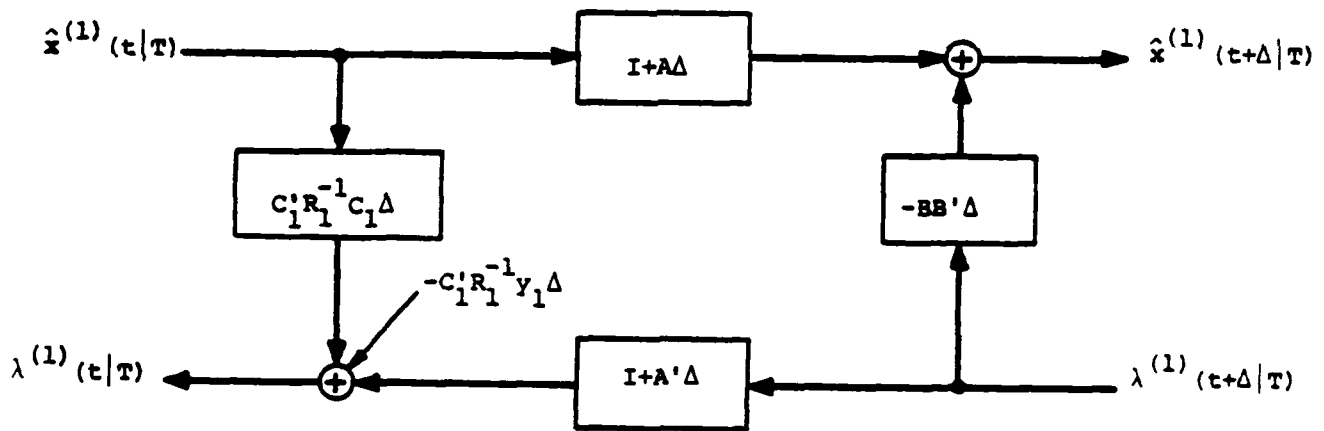


Figure 4.1b.    Local scattering layers for i=1.
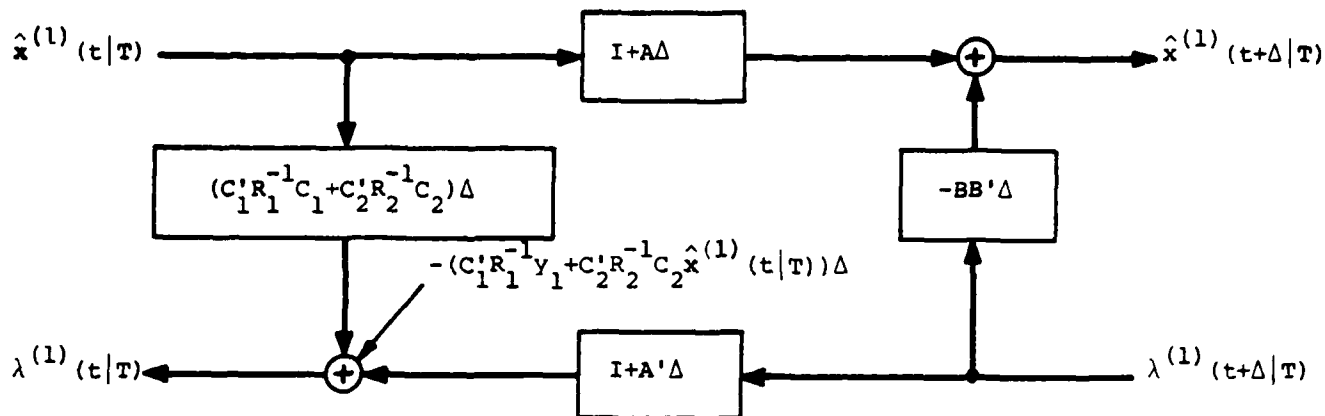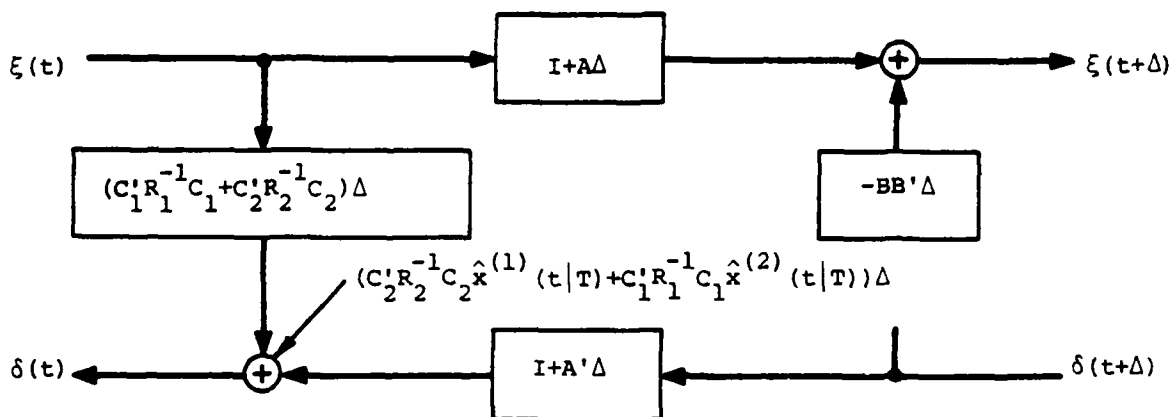
**Figure 4.1c.** Modification of the local scattering layers.



**Figure 4.2a.** Scattering layers obtained by superposition.

$$\hat{x}^{(1)}(t|T) \quad\quad \boxed{I+A\Delta} \quad\quad \oplus \quad\quad \hat{x}^{(1)}(t+\Delta|T)$$

$$\boxed{(C_1'R_1^{-1}C_1+C_2'R_2^{-1}C_2)\Delta} \quad\quad \boxed{-BB'\Delta}$$

$$-(C_1'R_1^{-1}y_1+C_2'R_2^{-1}C_2\hat{x}^{(1)}(t|T))\Delta$$

$$\lambda^{(1)}(t|T) \quad\quad \oplus \quad\quad \boxed{I+A'\Delta} \quad\quad \lambda^{(1)}(t+\Delta|T)$$

**Figure 4.1c.** Modification of the local scattering layers.

$$\xi(t) \quad\quad \boxed{I+A\Delta} \quad\quad \oplus \quad\quad \xi(t+\Delta)$$

$$\boxed{(C_1'R_1^{-1}C_1+C_2'R_2^{-1}C_2)\Delta} \quad\quad \boxed{-BB'\Delta}$$

$$(C_2'R_2^{-1}C_2\hat{x}^{(1)}(t|T)+C_1'R_1^{-1}C_1\hat{x}^{(2)}(t|T))\Delta$$

$$\delta(t) \quad\quad \oplus \quad\quad \boxed{I+A'\Delta} \quad\quad \delta(t+\Delta)$$
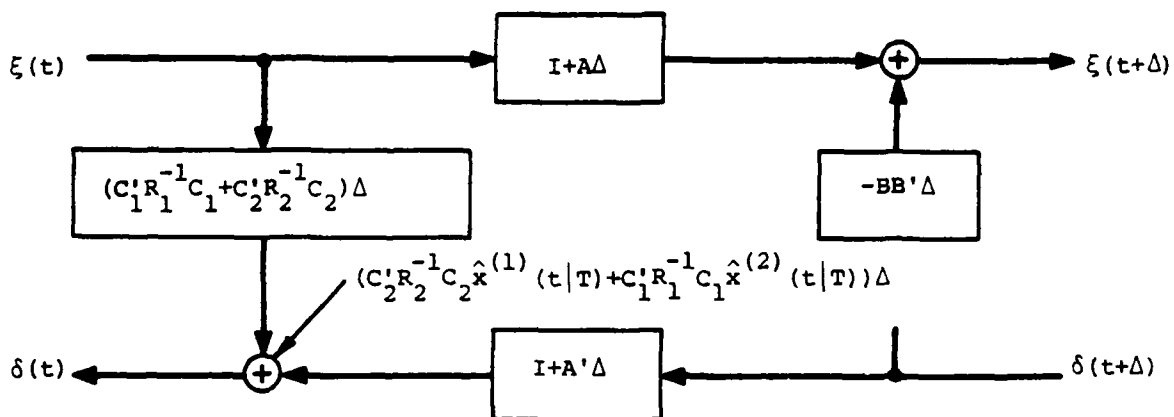
**Figure 4.2a.** Scattering layers obtained by superposition.

By symmetry, we can also transform the layers associated to $\hat{x}^{(2)}(t|T)$. Then, by superposition, we see in Figure 4.2a that the internal variables

$$\xi(t) = \hat{x}(t|T) - \hat{x}^{(1)}(t|T) - \hat{x}^{(2)}(t|T)$$

$$\delta(t) = \lambda(t|T) - \lambda^{(1)}(t|T) - \lambda^{(2)}(t|T)$$

$\hspace{4cm}$ (4.6)

can be obtained by composing the global scattering layers with internal sources

$$\tilde{q}(t)\Delta = \begin{pmatrix} 0 \\ C_2'R_2^{-1}C_2\hat{x}^{(1)}(t|T) + C_1'R_1^{-1}C_1\hat{x}^{(2)}(t|T) \end{pmatrix}\Delta \hspace{1cm} (4.7)$$

where $\tilde{q}(t)\Delta$ is obtained by superposing the sources of the global layers with the ones obtained in Figure 4.1c for the local layers. The boundary conditions for this scattering problem are also obtained by superposition and are given by

$$\xi(0) = -\hat{x}(0) - P(0)\delta(0)$$

$$\delta(T) = 0 \hspace{1cm} \cdot$$

$\hspace{4cm}$ (4.8)

They are described in Figure 4.2b where we have represented the aggregate scattering medium associated to $\xi(t)$ and $\delta(t)$.

To solve the scattering problem satisfied by $\xi(t)$, we can use any of the global smoothing algorithms with the observations $(y_1(t), y_2(t))$ replaced by $- (C_1\hat{x}^{(2)}(t|T), C_2\hat{x}^{(1)}(t|T))$ and with initial conditions $(-\hat{x}(0), P(0))$. The two-filter solution of this problem is

By symmetry, we can also transform the layers associated to $\hat{x}^{(2)}(t|T)$. Then, by superposition, we see in Figure 4.2a that the internal variables

$$\xi(t) = \hat{x}(t|T) - \hat{x}^{(1)}(t|T) - \hat{x}^{(2)}(t|T)$$

$$\delta(t) = \lambda(t|T) - \lambda^{(1)}(t|T) - \lambda^{(2)}(t|T)$$

(4.6)

can be obtained by composing the global scattering layers with internal sources

$$\tilde{q}(t)\Delta = \begin{pmatrix} 0 \\ C_2'R_2^{-1}C_2\hat{x}^{(1)}(t|T) + C_1'R_1^{-1}C_1\hat{x}^{(2)}(t|T) \end{pmatrix}\Delta \quad (4.7)$$

where $\tilde{q}(t)\Delta$ is obtained by superposing the sources of the global layers with the ones obtained in Figure 4.1c for the local layers. The boundary conditions for this scattering problem are also obtained by superposition and are given by

$$\xi(0) = -\hat{x}(0) - P(0)\delta(0)$$

$$\delta(T) = 0 \qquad \cdot$$

(4.8)

They are described in Figure 4.2b where we have represented the aggregate scattering medium associated to $\xi(t)$ and $\delta(t)$.

To solve the scattering problem satisfied by $\xi(t)$, we can use any of the global smoothing algorithms with the observations $(y_1(t), y_2(t))$ replaced by $- (C_1\hat{x}^{(2)}(t|T), C_2\hat{x}^{(1)}(t|T))$ and with initial conditions $(-\hat{x}(0), P(0))$. The two-filter solution of this problem is

Figure 4.2b.    Aggregate medium obtained by superposition.
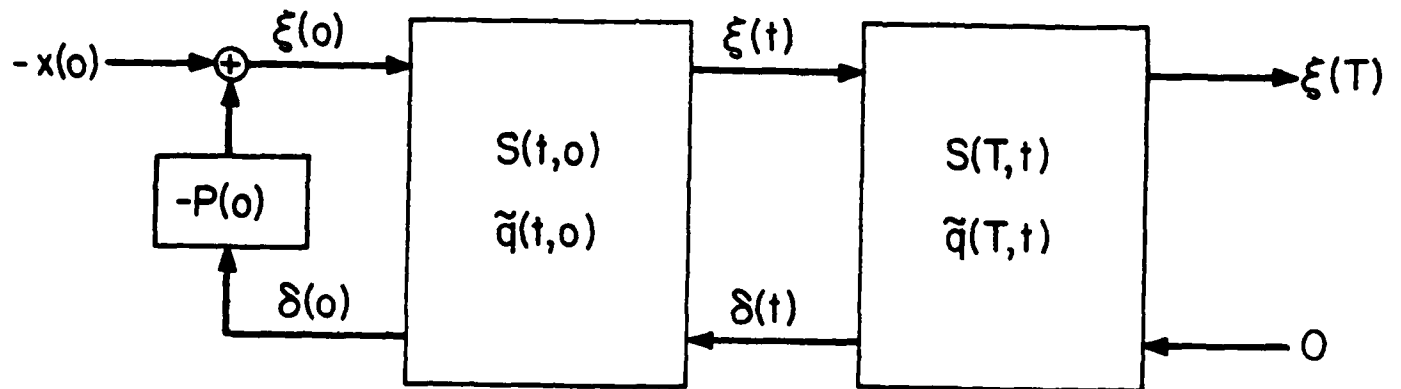
Figure 4.2b.   Aggregate medium obtained by superposition.

$$P_S^{-1}(t)\xi(t) = P_F^{-1}(t)\xi_F(t) + P_B^{-1}(t)\xi_B(t) \tag{4.9}$$

where

$$\dot{\xi}_F(t) = (A-P_F(C_1'R_1^{-1}C_1+C_2'R^{-1}C_2))\xi_F(t)$$
$$-P_F(C_1'R_1^{-1}C_1\hat{x}^{(2)}(t|T)+C_2'R_2^{-1}C_2\hat{x}^{(1)}(t|T)) \tag{4.10}$$
$$\xi_F(0) = -\hat{x}(0)$$

and

$$-\dot{\xi}_B(t) = (-A-P_B(C_1'R_1^{-1}C_1+C_2'R_2^{-1}C_2))\xi_B(t)$$
$$-P_B(C_1'R_1^{-1}C_1\hat{x}^{(2)}(t|T)+C_2'R_2^{-1}C_2\hat{x}^{(1)}(t|T)) \tag{4.11}$$
$$P_B^{-1}(T)\xi_B(T)=0$$

and where $P_S$, $P_F$ and $P_B$ denote respectively the global smoothing and forward and backward filtering error covariances. They are given by

$$P_S^{-1}(t) = P_F^{-1}(t) + P_B^{-1}(t) \tag{4.12}$$

with

$$\dot{P}_F(t) = AP_F+P_FA'+BB'-P_F(C_1'R_1^{-1}C_1+C_2'R_2^{-1}C_2)P_F \tag{4.13}$$
$$P_F(0) = P(0)$$

and

$$-\dot{P}_B(t) = -AP_B-P_BA'+BB'-P_B(C_1'R_1^{-1}C_1+C_2'R_2^{-1}C_2)P_B \tag{4.14}$$
$$P_B^{-1}(T) = 0.$$

Since the equations (4.9)-(4.11) for $\xi(t)$ depend only on the local smoothed estimates $\hat{x}^{(1)}(t|T)$ and $\hat{x}^{(2)}(t|T)$, and not on the measurements $y_1(\cdot)$ and $y_2(\cdot)$, they can be used by the central processor to compute the centralized estimate

$$\hat{x}(t|T) = \xi(t) + \hat{x}^{(1)}(t|T) + \hat{x}^{(2)}(t|T).$$

$$P_S^{-1}(t)\xi(t) = P_F^{-1}(t)\xi_F(t) + P_B^{-1}(t)\xi_B(t) \tag{4.9}$$

where

$$\dot{\xi}_F(t) = (A - P_F(C_1'R_1^{-1}C_1 + C_2'R^{-1}C_2))\xi_F(t)$$
$$- P_F(C_1'R_1^{-1}C_1\hat{x}^{(2)}(t|T) + C_2'R_2^{-1}C_2\hat{x}^{(1)}(t|T)) \tag{4.10}$$
$$\xi_F(0) = -\hat{x}(0)$$

and

$$-\dot{\xi}_B(t) = (-A - P_B(C_1'R_1^{-1}C_1 + C_2'R_2^{-1}C_2))\xi_B(t)$$
$$- P_B(C_1'R_1^{-1}C_1\hat{x}^{(2)}(t|T) + C_2'R_2^{-1}C_2\hat{x}^{(1)}(t|T)) \tag{4.11}$$
$$P_B^{-1}(T)\xi_B(T) = 0$$

and where $P_S$, $P_F$ and $P_B$ denote respectively the global smoothing and forward and backward filtering error covariances. They are given by

$$P_S^{-1}(t) = P_F^{-1}(t) + P_B^{-1}(t) \tag{4.12}$$

with

$$\dot{P}_F(t) = AP_F + P_F A' + BB' - P_F(C_1'R_1^{-1}C_1 + C_2'R_2^{-1}C_2)P_F \tag{4.13}$$
$$P_F(0) = P(0)$$

and

$$-\dot{P}_B(t) = -AP_B - P_B A' + BB' - P_B(C_1'R_1^{-1}C_1 + C_2'R_2^{-1}C_2)P_B \tag{4.14}$$
$$P_B^{-1}(T) = 0.$$

Since the equations (4.9)-(4.11) for $\xi(t)$ depend only on the local smoothed estimates $\hat{x}^{(1)}(t|T)$ and $\hat{x}^{(2)}(t|T)$, and not on the measurements $y_1(\cdot)$ and $y_2(\cdot)$, they can be used by the central processor to compute the centralized estimate

$$\hat{x}(t|T) = \xi(t) + \hat{x}^{(1)}(t|T) + \hat{x}^{(2)}(t|T).$$

This algorithm was first derived by purely algebraic methods in
[11].  One interesting application of this result is for the map
combining problem in the study of random fields.  In such a problem, we
consider a random field (gravitational, magnetic, or else) which is
modeled by the process  $x(t)$  considered above, and we are given two maps
$\hat{x}^{(i)}(t|T)$ $i=1,2$  obtained after two different surveys of this field.
Then, one wants to combine these maps to obtain a global map  $\hat{x}(t|T)$.
This can be done by using the previous algorithm (see [11] for more
details).

B.  <u>Decentralized filtering</u>

To obtain some similar formulas for the filtering estimates $\hat{x}(t|t)$,
$\hat{x}^{(1)}(t|t)$ and $\hat{x}^{(2)}(t|t)$, we need to apply the superposition principle to
the transmission layers associated to the filtering problem in information
filter form.  Let $P(t)$ and $P^{(i)}(t)$ $i=1,2$ be the solution of the Riccati
equations

$$\dot{P} = AP+PA'-BB'-P(C_1'R_1^{-1}C_1+C_2'R_2^{-1}C_2)P \tag{4.15}$$

and

$$\dot{P}^{(i)} = AP^{(i)}+P^{(i)}A'+BB'-P^{(i)}C_i'R_i^{-1}C_iP^{(i)} \tag{4.16}$$

with   $P^{(i)}(0) = P(0)$ $i=1,2$.  Then if

$$d(t) = P^{-1}(t)\hat{x}(t) \quad \text{and} \quad d^{(i)}(t) = P^{(i)-1}\hat{x}^{(i)}(t) \tag{4.17}$$

denote the global and local filtered estimates of $x(t)$ in information
form, the transmission layers associated to $d$ and  $d^{(i)}$ are

This algorithm was first derived by purely algebraic methods in [11]. One interesting application of this result is for the map combining problem in the study of random fields. In such a problem, we consider a random field (gravitational, magnetic, or else) which is modeled by the process $x(t)$ considered above, and we are given two maps $\hat{x}^{(i)}(t|T)$ $i=1,2$ obtained after two different surveys of this field. Then, one wants to combine these maps to obtain a global map $\hat{x}(t|T)$. This can be done by using the previous algorithm (see [11] for more details).

## B. Decentralized filtering

To obtain some similar formulas for the filtering estimates $\hat{x}(t|t)$, $\hat{x}^{(1)}(t|t)$ and $\hat{x}^{(2)}(t|t)$, we need to apply the superposition principle to the transmission layers associated to the filtering problem in information filter form. Let $P(t)$ and $P^{(i)}(t)$ $i=1,2$ be the solution of the Riccati equations

$$\dot{P} = AP+PA'+BB'-P(C_1'R_1^{-1}C_1+C_2'R_2^{-1}C_2)P \qquad (4.15)$$

and

$$\dot{P}^{(i)} = AP^{(i)}+P^{(i)}A'+BB'-P^{(i)}C_i'R_i^{-1}C_iP^{(i)} \qquad (4.16)$$

with $P^{(i)}(0) = P(0)$ $i=1,2$. Then if

$$d(t) = P^{-1}(t)\hat{x}(t) \quad \text{and} \quad d^{(i)}(t) = P^{(i)-1}\hat{x}^{(i)}(t) \qquad (4.17)$$

denote the global and local filtered estimates of $x(t)$ in information form, the transmission layers associated to $d$ and $d^{(i)}$ are

described in Figures 4.3a and b. These media are not the same, but by adding and subtracting to the local transmission matrices, we can modify the local layers as shown in Figure 4.3c, so that the transmission media are now identical. By superposition, we find in Figure 4.4 that

$$l(t) = d(t) - d^{(1)}(t) - d^{(2)}(t) \tag{4.18}$$

satisfies a transmission problem with the same medium as the global filtering problem and with sources

$$\tilde{q}(t)\Delta = ((P^{(1)-1}-P^{-1})BB'd^{(1)}+(P^{(2)-1}-P^{-1})BB'd^{(2)})\Delta . \tag{4.19}$$

Thus, $l(t)$ is given by

$$l(t) = (A'+P^{-1}BB')l(t)$$
$$+((P^{(1)-1}-P^{-1})BB'd^{(1)}+(P^{(2)-1}-P^{-1})BB'd^{(2)}) \tag{4.20}$$

with initial conditions

$$l(0) = -d(0) = -P^{-1}(0)\hat{x}(0) . \tag{4.21}$$

If instead of computing $l(t)$, we want to compute

$$r(t) = P(t)l(t) = \hat{x}(t) - P(t)(P^{(1)-1}\hat{x}^{(1)}+P^{(2)-1}\hat{x}^{(2)}) \tag{4.22}$$

we find that

$$\dot{r}(t) = (A-P(C_1'R_1^{-1}C_1+C_2'R_2^{-1}C_2))r(t)$$
$$+ (PP^{(1)-1}-I)BB'P^{(1)-1}\hat{x}^{(1)}(t)+(PP^{(2)-1}-I)BB'P^{(2)-1}\hat{x}^{(2)}(t) \tag{4.23}$$

described in Figures 4.3a and b. These media are not the same, but by adding and subtracting to the local transmission matrices, we can modify the local layers as shown in Figure 4.3c, so that the transmission media are now identical. By superposition, we find in Figure 4.4 that

$$l(t) = d(t) - d^{(1)}(t) - d^{(2)}(t) \tag{4.18}$$

satisfies a transmission problem with the same medium as the global filtering problem and with sources

$$\tilde{q}(t)\Delta = ((P^{(1)-1} - P^{-1})BB'd^{(1)} + (P^{(2)-1} - P^{-1})BB'd^{(2)})\Delta . \tag{4.19}$$

Thus, $l(t)$ is given by

$$l(t) = (A' + P^{-1}BB')l(t) \\ + ((P^{(1)-1} - P^{-1})BB'd^{(1)} + (P^{(2)-1} - P^{-1})BB'd^{(2)}) \tag{4.20}$$

with initial conditions

$$l(0) = -d(0) = -P^{-1}(0)\hat{x}(0) . \tag{4.21}$$

If instead of computing $l(t)$, we want to compute

$$r(t) = P(t)l(t) = \hat{x}(t) - P(t)(P^{(1)-1}\hat{x}^{(1)} + P^{(2)-1}\hat{x}^{(2)}) \tag{4.22}$$

we find that

$$\dot{r}(t) = (A - P(C_1'R_1^{-1}C_1 + C_2'R_2^{-1}C_2))r(t) \\ + (PP^{(1)-1} - I)BB'P^{(1)-1}\hat{x}^{(1)}(t) + (PP^{(2)-1} - I)BB'P^{(2)-1}\hat{x}^{(2)}(t) \tag{4.23}$$

$$d(t) \longrightarrow \boxed{I-(A'+P^{-1}BB')\Delta} \longrightarrow \oplus \longrightarrow d(t+\Delta)$$

$$(C_1'R_1^{-1}y_1+C'R_2^{-1}y_2)\Delta$$

(a)

$$d^{(i)}(t) \longrightarrow \boxed{I-(A'+P^{(i)-1}BB')\Delta} \longrightarrow \oplus \longrightarrow d^{(i)}(t+\Delta)$$

$$C_i'R_i^{-1}y_i\Delta$$

(b)

$$d^{(i)}(t) \longrightarrow \boxed{I-(A'+P^{(i)-1}BB')\Delta} \longrightarrow \oplus \longrightarrow d^{(i)}(t+\Delta)$$

$$(C_i'R_i^{-1}y_i+(P^{-1}-P^{(i)-1})BB'd^{(i)})\Delta$$

(c)

**Figure 4.3.** (a) Global transmission layers, (b) local Transmission layers, (c) Modification of the local transmission layers.

$$d(t) \longrightarrow \boxed{I-(A'+P^{-1}BB')\Delta} \longrightarrow \oplus \longrightarrow d(t+\Delta)$$

$$(C_1'R_1^{-1}y_1+C'R_2^{-1}y_2)\Delta$$

(a)

$$d^{(i)}(t) \longrightarrow \boxed{I-(A'+P^{(i)-1}BB')\Delta} \longrightarrow \oplus \longrightarrow d^{(i)}(t+\Delta)$$

$$C_i'R_i^{-1}y_i\Delta$$

(b)

$$d^{(i)}(t) \longrightarrow \boxed{I-(A'+P^{(i)-1}BB')\Delta} \longrightarrow \oplus \longrightarrow d^{(i)}(t+\Delta)$$

$$(C_i'R_i^{-1}y_i+(P^{-1}-P^{(i)-1})BB'd^{(i)})\Delta$$

(c)

**Figure 4.3.** (a) Global transmission layers, (b) local Transmission layers, (c) Modification of the local transmission layers.

$$I(t) \longrightarrow \boxed{I - (A' + P^{-1}BB')\Delta} \longrightarrow \oplus \longrightarrow I(t + \Delta)$$

$$((P^{(1)-1}-P^{-1})BB'd^{(1)}+(P^{(2)-1}-P^{-1})BB'd^{(2)})\Delta$$

Figure 4.4.   Transmission layers obtained by superposition.



Central Processor

Computation of r

$$\hat{x} = r + P(P^{(1)-1}\hat{x}^{(1)} + P^{(2)-1}\hat{x}^{(2)})$$

$\hat{x}^{(1)}$

$\hat{x}^{(2)}$

Local Processor 1:

Kalman filter for $\hat{x}^{(1)}$

Local Processor 2:

Kalman filter for $\hat{x}^{(2)}$

$Y_1$

$Y_2$

Figure 4.5.   Centralized implementation of the decentralized filtering algorithm.

$$I(t) \longrightarrow \boxed{I - (A' + P^{-1}BB')\Delta} \longrightarrow \oplus \longrightarrow I(t + \Delta)$$

$$((P^{(1)-1} - P^{-1})BB'd^{(1)} + (P^{(2)-1} - P^{-1})BB'd^{(2)})\Delta$$

Figure 4.4.  Transmission layers obtained by superposition.

Central Processor

Computation of r

$$\hat{x} = r + P(P^{(1)-1}\hat{x}^{(1)} + P^{(2)-1}\hat{x}^{(2)})$$

$\hat{x}^{(1)}$

$\hat{x}^{(2)}$

Local Processor 1:

Kalman filter for $\hat{x}^{(1)}$

Local Processor 2:

Kalman filter for $\hat{x}^{(2)}$

$Y_1$

$Y_2$
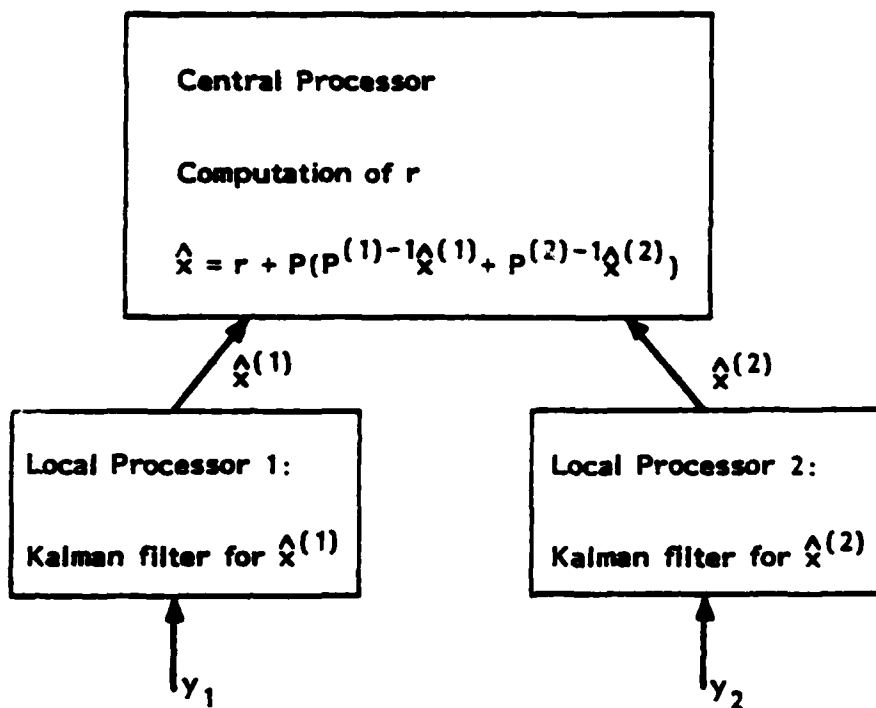
Figure 4.5.  Centralized implementation of the decentralized
filtering algorithm.

with $r(0) = -\hat{x}(0)$, an algorithm which is the same as the one obtained algebraically by Speyer [9] and Willsky et.al. [11].

## Implementation

The previous decentralized filtering algorithm can be implemented in two ways, depending on whether $l(t)$ or $r(t)$ are computed in a centralized or decentralized way, as shown in Figures 4.5 and 4.6 (see also Chong [10]).

In the centralized method, $r(t)$ is computed by the central processor, and in the decentralized method the local processor $S_i$ computes

$$\dot{r}^{(i)}(t) = (A - P(C_1'R_1^{-1}C_1 + C_2'R_2^{-1}C_2))r^{(i)}(t) \qquad (4.24)$$

$$+ (PP^{(i)-1} - I)BB'P^{(i)-1}\hat{x}^{(i)}(t)$$

with i=1,2. The initial conditions can be chosen such that

$$r^{(1)}(0) = -x(0), \quad r^{(2)}(0) = 0 . \qquad (4.25)$$

Then, by linearity, one gets

$$r(t) = r^{(1)}(t) + r^{(2)}(t) \qquad (4.26)$$

so that if each local processor transmits $(\hat{x}^{(i)}, r^{(i)})$, the central estimate $x$ is obtained by a static linear combination of the $\hat{x}^{(i)}$'s and $r^{(i)}$'s, i.e.

$$\hat{x} = r^{(1)} + r^{(2)} + P(P^{(1)-1}\hat{x}^{(1)} + P^{(2)-1}\hat{x}^{(2)}) . \qquad (4.27)$$

with $r(0) = -\hat{x}(0)$, an algorithm which is the same as the one obtained algebraically by Speyer [9] and Willsky et.al. [11].

## Implementation

The previous decentralized filtering algorithm can be implemented in two ways, depending on whether $l(t)$ or $r(t)$ are computed in a centralized or decentralized way, as shown in Figures 4.5 and 4.6 (see also Chong [10]).

In the centralized method, $r(t)$ is computed by the central processor, and in the decentralized method the local processor $S_i$ computes

$$\dot{r}^{(i)}(t) = (A - P(C_1'R_1^{-1}C_1 + C_2'R_2^{-1}C_2))r^{(i)}(t) \qquad (4.24)$$

$$+ (PP^{(i)-1} - I)BB'P^{(i)-1}\hat{x}^{(i)}(t)$$

with $i=1,2$. The initial conditions can be chosen such that

$$r^{(1)}(0) = -x(0), \quad r^{(2)}(0) = 0 . \qquad (4.25)$$

Then, by linearity, one gets

$$r(t) = r^{(1)}(t) + r^{(2)}(t) \qquad (4.26)$$

so that if each local processor transmits $(\hat{x}^{(i)}, r^{(i)})$, the central estimate $x$ is obtained by a static linear combination of the $\hat{x}^{(i)}$'s and $r^{(i)}$'s, i.e.

$$\hat{x} = r^{(1)} + r^{(2)} + P(P^{(1)-1}\hat{x}^{(1)} + P^{(2)-1}\hat{x}^{(2)}) . \qquad (4.27)$$

**Figure 4.6.** Decentralized implementation of the decentralized filtering algorithm.

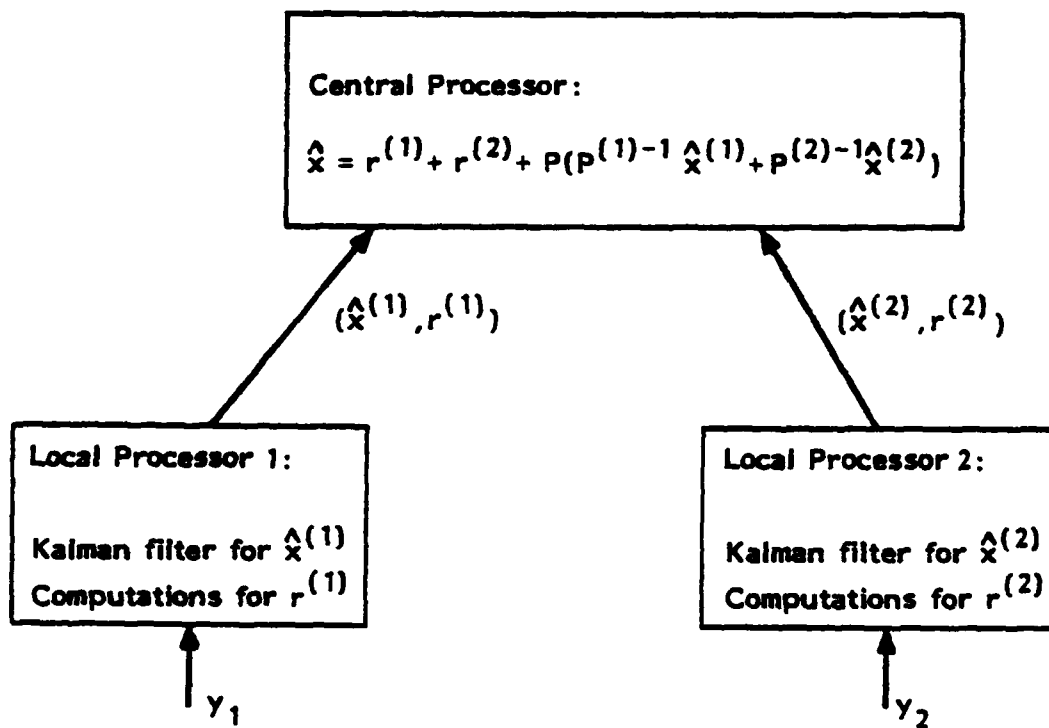**Figure 4.6.** Decentralized implementation of the decentralized filtering algorithm.

The advantage of such a procedure is that if the communications between

the local processor and the central one are restricted; and if the

central processor needs to compute $\hat{x}$ only from time to time, then the

local processors can send $(\hat{x}^{(i)}, r^{(i)})$ at these times instead of sending

. . .

The advantage of such a procedure is that if the communications between the local processor and the central one are restricted; and if the central processor needs to compute $\hat{x}$ only from time to time, then the local processors can send $(\hat{x}^{(i)}, r^{(i)})$ at these times instead of sending $\hat{x}^{(i)}$ continuously. However, a disadvantage of this method is that it assumes that each local processor knows completely the global model, i.e. $C_j$ and $R_j$ for $j \neq i$.

## V. The smoothing update and real-time smoothing problems

In some cases, instead of being given the local smoothed estimates $\hat{x}^{(i)}(t|T)$ $i=1,2$ to compute $\hat{x}(t|T)$, one is given a first estimate $\hat{x}^{(i)}(t|T)$ and some additional measurements $y_2(t)$, $0 \leq t \leq T$. Then, instead of reprocessing all the data $(y_1(\cdot), y_2(\cdot))$ to compute $\hat{x}(t|T)$, one is interested in updating the first estimate $\hat{x}^{(1)}(t|T)$ to incorporate the new measurements $y_2(\cdot)$. This problem will be called in the following the smoothing update problem. It arises for example in the analysis of random fields [11], when one wants to update an old map to incorporate the results of new surveys.

A variant of this problem that will also be discussed in the following is the real-time smoothing problem. In this case, if

$$Y_1 = H(m; y_1(s), 0 \leq s \leq T)$$

$$Y_2^t = H(m; y_2(s), 0 \leq s \leq t), \quad 0 \leq t \leq T$$

denote the Hilbert spaces spanned by all the observations $y_1(\cdot)$ and by the observations $y_2(\cdot)$ <u>up to time t</u>, one wants to compute

$$\hat{x}_{RS}(t) = E[x(t)|Y_1, Y_2^t] \tag{5.1}$$

in terms of $\hat{x}^{(1)}(t|T)$ and $y_2(\cdot)$.

## A. The smoothing update problem

By superposing the global and local scattering layers obtained for the decentralized smoothing problem, and by denoting

$$\eta(t) = \hat{x}(t|T) - \hat{x}^{(1)}(t|T) \tag{5.2}$$

$$\theta(t) = \lambda(t|T) - \lambda^{(1)}(t|T) \tag{5.3}$$

we obtain the scattering layers described in Figure 5.1. This shows that the variables $\eta(t)$ and $\theta(t)$ can be obtained by composing the global scattering layers with internal sources

$$\tilde{q}(t)\Delta = \begin{pmatrix} 0 \\ -C_2' R_2^{-1}(y_2 - C_2\hat{x}^{(1)}(t|T)) \end{pmatrix} \Delta . \tag{5.4}$$

The boundary conditions for this scattering problem can also be obtain by superposition, and they are given by

$$\eta(0) = -P(0)\theta(0), \quad \theta(T) = 0 . \tag{5.5}$$

To solve this scattering problem, we can use again the two-filter formula. We obtain

$$P_s^{-1}(t)\eta(t) = P_F^{-1}(t)\eta_F(t) + P_B^{-1}(t)\eta_B(t) \tag{5.6}$$

**Figure 5.1.** Scattering layers associated to the smoothing update problem.



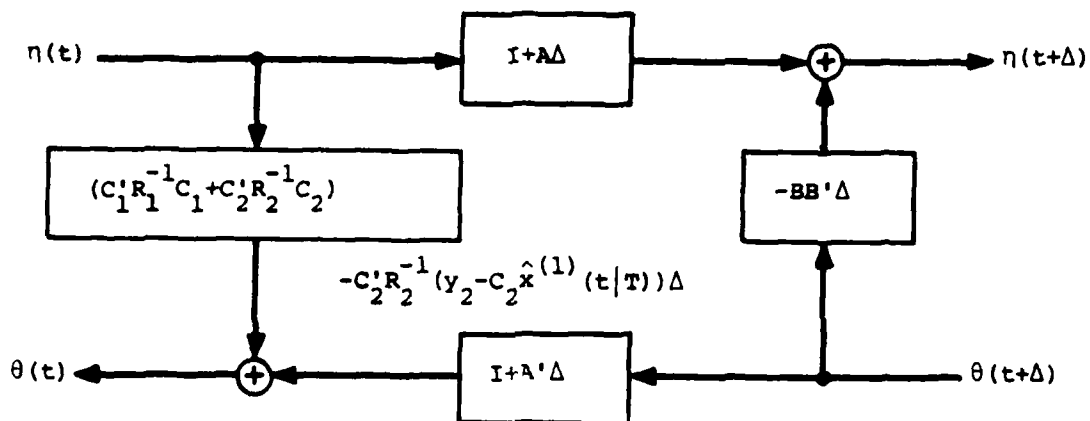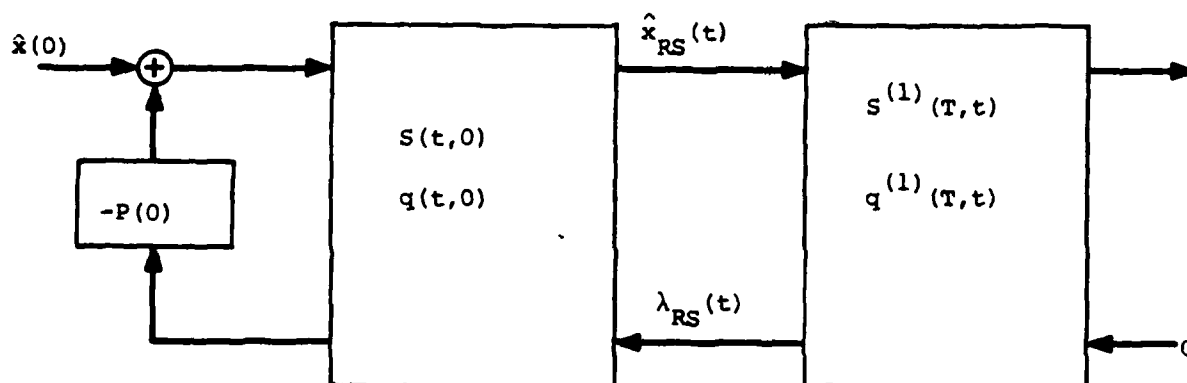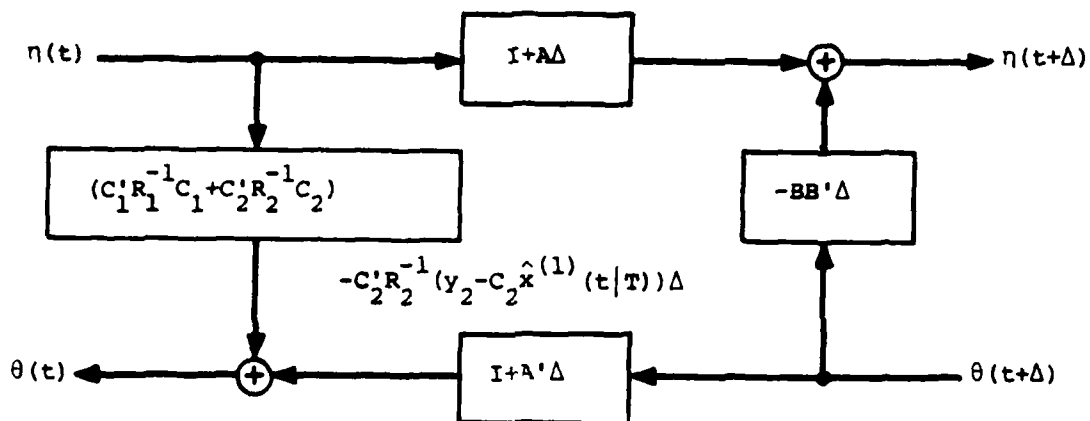**Figure 5.2a.** Scattering medium for the real-time smoothing problem.

Figure 5.1. Scattering layers associated to the smoothing update problem.
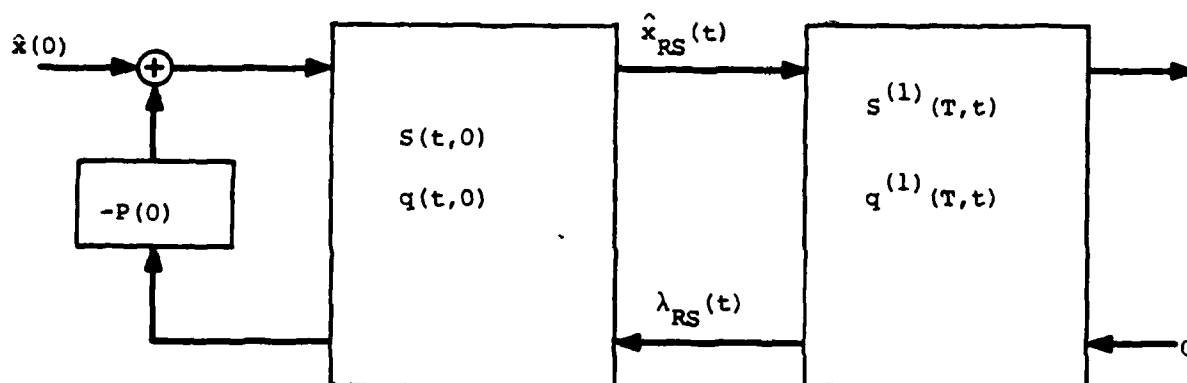


Figure 5.2a. Scattering medium for the real-time smoothing problem.

where

$$\dot{\eta}_F(t) = (A - P_F(C_1'R_1^{-1}C_1 + C_2'R_2^{-1}C_2))\eta_F(t)$$ 

$$+ P_F C_2' R_2^{-1}(y_2(t) - C_2\hat{x}^{(1)}(t|T))$$

$$\eta_F(0) = 0$$

(5.7)

and

$$-\dot{\eta}_B(t) = (-A - P_B(C_1'R_1^{-1}C_1 + C_2'R_2^{-1}C_2))\eta_B(t)$$

$$+ P_B C_2' R_2^{-1}(y_2(t) - C_2\hat{x}^{(1)}(t|T))$$

$$P_B^{-1}(T)\eta_B(T) = 0 ,$$

(5.8)

and where $P_S$, $P_F$ and $P_B$ are given by (4.12)-(4.14).


## B.  The real-time smoothing problem

The aggregate scattering medium associated to the real-time smoothing problem is given by Figure 5.2a where the scattering matrix $S(t,0)$ and the sources $q(t,0)$ are generated by the global scattering layers described in Figure 4.1a and where the medium $S^{(1)}(T,t)$ and the sources $q^{(1)}(T,t)$ are generated by the local layers of Figure 4.1b.

Similarly, the local scattering medium associated to $\hat{x}^{(1)}(t|T)$ is given by Figure 5.2b, where the right layer is the same as for $\hat{x}_{RS}(t)$. The medium of the left layer can be made the same as for $\hat{x}_{RS}(t)$ if we use the modification of the local layers described in Figure 4.1c. By doing so, we find that

where

$$\dot{\eta}_F(t) = (A - P_F(C_1'R_1^{-1}C_1 + C_2'R_2^{-1}C_2))\eta_F(t) \qquad (5.7)$$

$$+ P_F C_2' R_2^{-1}(y_2(t) - C_2\hat{x}^{(1)}(t|T))$$

$$\eta_F(0) = 0$$

and

$$-\dot{\eta}_B(t) = (-A - P_B(C_1'R_1^{-1}C_1 + C_2'R_2^{-1}C_2))\eta_B(t) \qquad (5.8)$$

$$+ P_B C_2' R_2^{-1}(y_2(t) - C_2\hat{x}^{(1)}(t|T))$$

$$P_B^{-1}(T)\eta_B(T) = 0 \ ,$$

and where $P_S$, $P_F$ and $P_B$ are given by (4.12)-(4.14).


## B. The real-time smoothing problem

The aggregate scattering medium associated to the real-time smoothing problem is given by Figure 5.2a where the scattering matrix $S(t,0)$ and the sources $q(t,0)$ are generated by the global scattering layers described in Figure 4.1a and where the medium $S^{(1)}(T,t)$ and the sources $q^{(1)}(T,t)$ are generated by the local layers of Figure 4.1b.

Similarly, the local scattering medium associated to $\hat{x}^{(1)}(t|T)$ is given by Figure 5.2b, where the right layer is the same as for $\hat{x}_{RS}(t)$. The medium of the left layer can be made the same as for $\hat{x}_{RS}(t)$ if we use the modification of the local layers described in Figure 4.1c. By doing so, we find that

Figure 5.2b.  Local scattering medium.
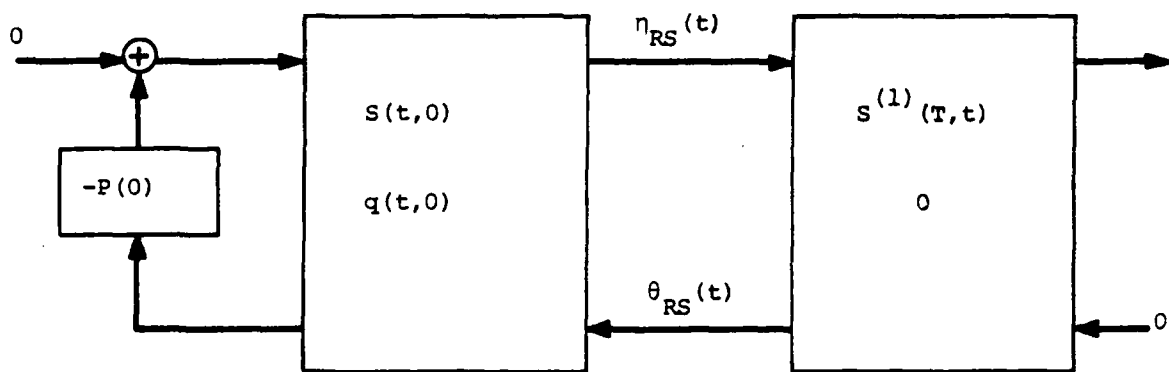


Figure 5.2c.  Scattering problem obtained by superposition.
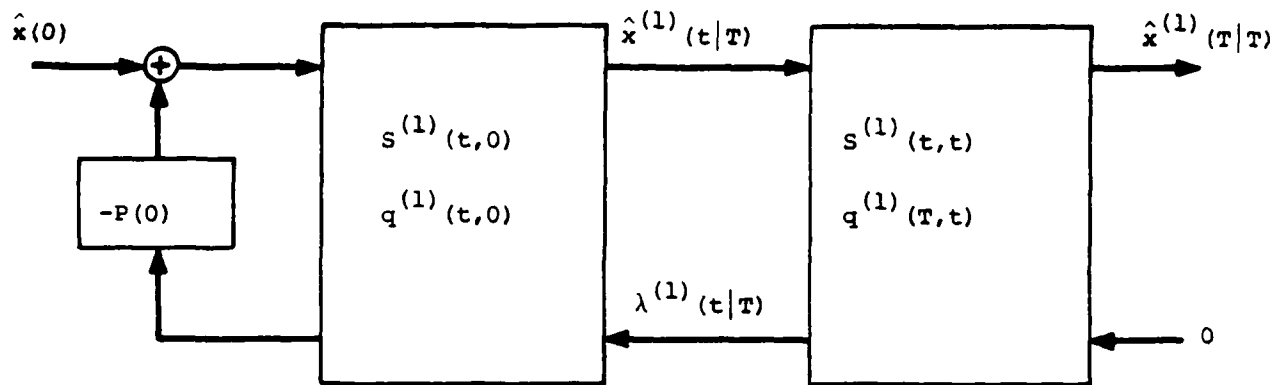
Figure 5.2b.    Local scattering medium.
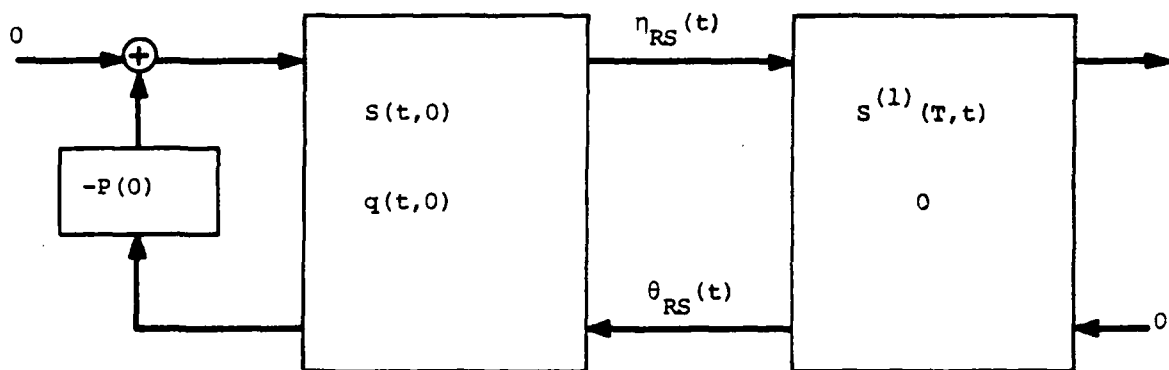


Figure 5.2c.    Scattering problem obtained by superposition.

and

$$P_{RS}^{-1}(t) = P_F^{-1}(t) + P_B^{(1)-1}(t) \cdot \tag{5.15}$$

From (5.13) one get $\eta_B^{(1)}(t) \equiv 0$, so that (5.12) reduces to

$$\hat{x}_{RS}(t) = \hat{x}^{(1)}(t|T) + P_{RS}P_F^{-1}\eta_F(t) \cdot \tag{5.16}$$

A similar problem to those discussed here appears when one considers a decentralized filtering structure where instead of assuming

$$\begin{pmatrix} \dot{x}_1 \\ \dot{s}_1 \end{pmatrix} = \underbrace{\begin{pmatrix} A_1 & 0 \\ A_{21} & A_{22} \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ s_1 \end{pmatrix} + \underbrace{\begin{pmatrix} B_1 \\ B_{21} \end{pmatrix}}_{B} u \qquad (6.4)$$

and

$$y_1 = \underbrace{\begin{pmatrix} H_1 & 0 \end{pmatrix}}_{C_1} \begin{pmatrix} x_1 \\ s_1 \end{pmatrix} + v_1 \qquad (6.5a)$$

$$y_2 = \underbrace{\begin{pmatrix} C_{21} & C_{22} \end{pmatrix}}_{C_2} \begin{pmatrix} x_1 \\ s_2 \end{pmatrix} + v_2 . \qquad (6.5b)$$

With this model structure, the smoothed estimates $\hat{x}(t|T)$ and $\hat{x}_1^{(1)}(t|T)$ can be obtained by composing the global and local scattering layers described in Figures 6.1a and 6.1b. In Figure 6.1a, it is assumed that the matrices $A, B, C_1$ and $C_2$ have the same block structure as in (6.4)-(6.5). The main feature of the local scattering layers is that the internal variables $\hat{x}_1^{(1)}(t|T)$ and $\lambda_1^{(1)}(t|T)$ are of smaller dimension than the variables

$$\hat{x}(t|T) = \begin{pmatrix} \hat{x}_1(t|T) \\ \hat{s}_1(t|T) \end{pmatrix} , \quad \lambda(t|T) = \begin{pmatrix} \lambda_1(t|T) \\ \sigma_1(t|T) \end{pmatrix} \qquad (6.6)$$

associated to the global scattering problem. However, by adding some appropriate internal sources, the local scattering layers can be imbedded in the global ones as shown in Figure 6.1c and the internal variables

simplified models which describe only the evolution of a set of

local variables (machine angles and frequencies). This motivates us

in this section to consider the case when the global model is given by

(4.1) and (4.2) and when the local models and measurements are given

by

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t)$$
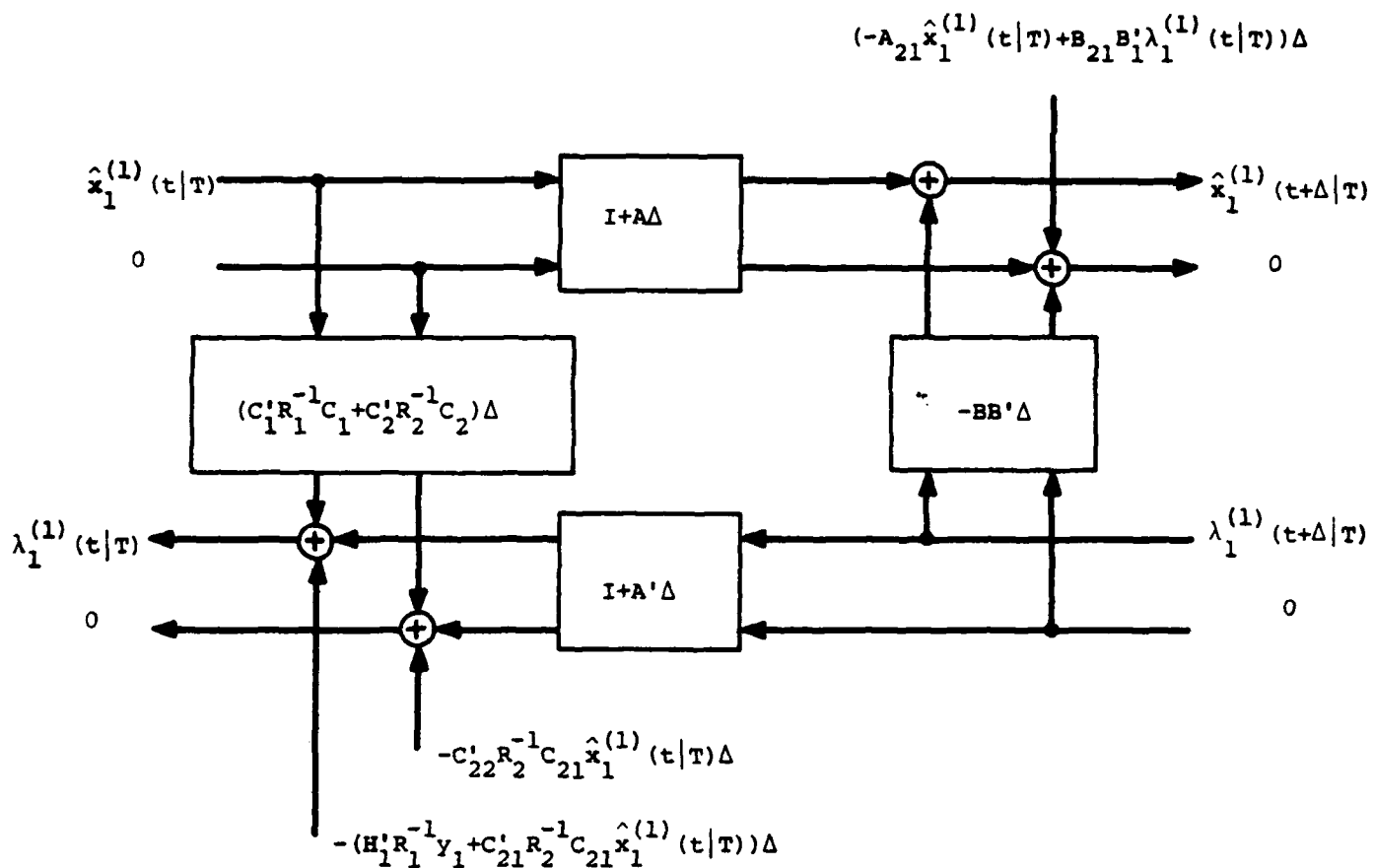
Figure 6.1c.   Imbedding of the local layers.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{s}_1 \end{pmatrix} = \underbrace{\begin{pmatrix} A_1 & 0 \\ A_{21} & A_{22} \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ s_1 \end{pmatrix} + \underbrace{\begin{pmatrix} B_1 \\ B_{21} \end{pmatrix}}_{B} u \qquad (6.4)$$

and

$$y_1 = \underbrace{\begin{pmatrix} H_1 & 0 \end{pmatrix}}_{C_1} \begin{pmatrix} x_1 \\ s_1 \end{pmatrix} + v_1 \qquad (6.5a)$$

$$y_2 = \underbrace{\begin{pmatrix} C_{21} & C_{22} \end{pmatrix}}_{C_2} \begin{pmatrix} x_1 \\ s_2 \end{pmatrix} + v_2 . \qquad (6.5b)$$

With this model structure, the smoothed estimates $\hat{x}(t|T)$ and $\hat{x}_1^{(1)}(t|T)$ can be obtained by composing the global and local scattering layers described in Figures 6.1a and 6.1b. In Figure 6.1a, it is assumed that the matrices $A, B, C_1$ and $C_2$ have the same block structure as in (6.4)-(6.5). The main feature of the local scattering layers is that the internal variables $\hat{x}_1^{(1)}(t|T)$ and $\lambda_1^{(1)}(t|T)$ are of smaller dimension than the variables

$$\hat{x}(t|T) = \begin{pmatrix} \hat{x}_1(t|T) \\ \hat{s}_1(t|T) \end{pmatrix} , \quad \lambda(t|T) = \begin{pmatrix} \lambda_1(t|T) \\ \sigma_1(t|T) \end{pmatrix} \qquad (6.6)$$

associated to the global scattering problem. However, by adding some appropriate internal sources, the local scattering layers can be imbedded in the global ones as shown in Figure 6.1c and the internal variables

$$\begin{pmatrix} \hat{x}_1^{(1)}(t|T) \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \lambda_1^{(1)}(t|T) \\ 0 \end{pmatrix}$$

have now the same dimension as $\hat{x}(t|T)$ and $\lambda(t|T)$.

By superposing the global scattering layers of Figure 6.1a and the imbedded local layers of Figure 6.1c, we obtain the scattering
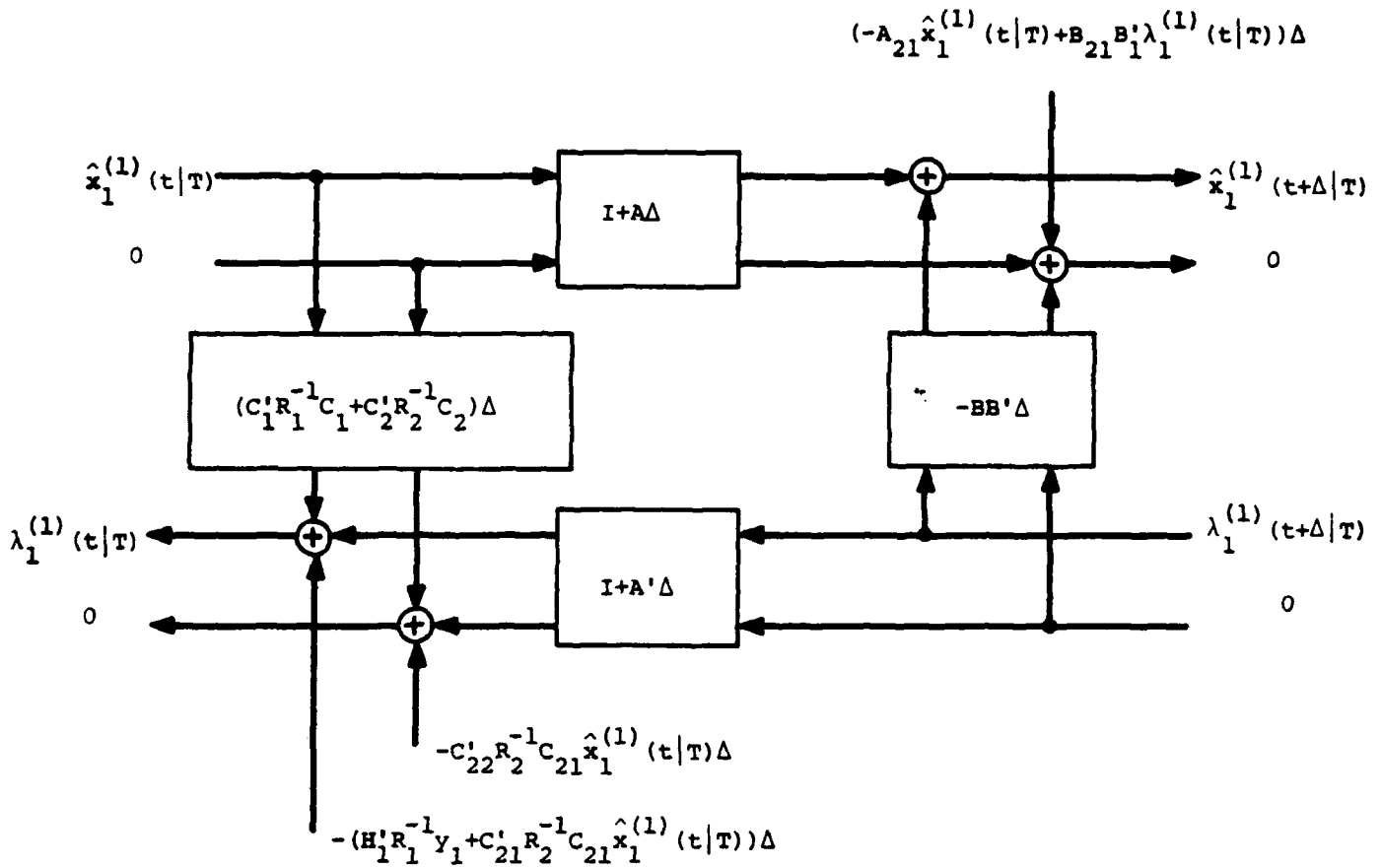
Figure 6.1c.   Imbedding of the local layers.

seek below to transform the problem so that $B_{21}B_1'=0$. When this is the case, the two-filter solution of the scattering problem is given by

$$P_s^{-1}(t) \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} = P_F^{-1}(t) \begin{pmatrix} \xi_F(t) \\ \eta_F(t) \end{pmatrix} + P_B^{-1}(t) \begin{pmatrix} \xi_B(t) \\ \eta_B(t) \end{pmatrix} \quad (6.9)$$

where

$$\begin{pmatrix} \dot{\xi}_F(t) \\ \dot{\eta}_F(t) \end{pmatrix} = \left(A - P_F(C_1'R_1^{-1}C_1 + C_2'R_2^{-1}C_2)\right) \begin{pmatrix} \xi_F(t) \\ \eta_F(t) \end{pmatrix} \quad (6.10)$$

$$+ P_F C_2'R_2^{-1}(y_2(t) - C_{21}\hat{x}_1^{(1)}(t|T)) + \begin{pmatrix} 0 \\ A_{21}\hat{x}_1^{(1)}(t|T) \end{pmatrix}$$

$$\begin{pmatrix} \xi_F(0) \\ \eta_F(0) \end{pmatrix} = \begin{pmatrix} 0 \\ \hat{s}_1(0) \end{pmatrix}$$

and

$$-\begin{pmatrix} \dot{\xi}_B(t) \\ \dot{\eta}_B(t) \end{pmatrix} = (-A - P_B(C_1'R_1^{-1}C_1 + C_2'R_2^{-1}C_2)) \begin{pmatrix} \xi_B(t) \\ \eta_B(t) \end{pmatrix} \quad (6.11)$$

$$+ P_B C_2'R_2^{-1}(y_2(t) - C_{21}\hat{x}_1^{(1)}(t|T)) - \begin{pmatrix} 0 \\ A_{21}\hat{x}_1^{(1)}(t|T) \end{pmatrix}$$

$$P_B^{-1}(T) \begin{pmatrix} \xi_B(T) \\ \eta_B(T) \end{pmatrix} = 0$$

$$\begin{pmatrix} \hat{x}_1^{(1)}(t|T) \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \lambda_1^{(1)}(t|T) \\ 0 \end{pmatrix}$$

have now the same dimension as $\hat{x}(t|T)$ and $\lambda(t|T)$.

By superposing the global scattering layers of Figure 6.1a and the imbedded local layers of Figure 6.1c, we obtain the scattering layers described in Figure 6.2. This shows that the internal variables

$$\begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} = \begin{pmatrix} \hat{x}_1(t|T) - \hat{x}_1^{(1)}(t|T) \\ \hat{s}_1(t|T) \end{pmatrix} \tag{6.7a}$$

and

$$\begin{pmatrix} \delta(t) \\ \theta(t) \end{pmatrix} = \begin{pmatrix} \lambda_1(t|T) - \lambda_1^{(1)}(t|T) \\ \sigma_1(t|T) \end{pmatrix} \tag{6.7b}$$

satisfy a scattering problem with the same medium as for the global smoothing problem and with internal sources

$$\tilde{q}_+(t)\Delta = \begin{pmatrix} 0 \\ A_{21}\hat{x}_1^{(1)}(t|T) - B_{21}B_1'\lambda_1^{(1)}(t|T) \end{pmatrix}\Delta \tag{6.8a}$$

and

$$\tilde{q}_-(t)\Delta = - \begin{pmatrix} C_{21}' \\ C_{22}' \end{pmatrix} R_2^{-1}(y_2(t) - C_{21}\hat{x}_1^{(1)}(t|T))\Delta \quad . \tag{6.8b}$$

These sources require the knowledge of $\lambda_1^{(1)}(t|T)$ as well as of $\hat{x}_1^{(1)}(t|T)$. This is not very convenient, and consequently we will

and where $P_S$, $P_F$ and $P_B$ are given by (4.12)-(4.14) with the special matrix structure (6.4)-(6.5). Since equations (6.10)-(6.11) depend only on the local estimates $\hat{x}_1^{(1)}(t|T)$ and on the new measurements $y_2(\cdot)$, we can use them to compute the centralized smoothed estimate

$$\hat{x}(t|T) = \begin{pmatrix} \hat{x}_1^{(1)}(t|T) \\ 0 \end{pmatrix} + P_S \left( P_F^{-1} \begin{pmatrix} \xi_F(t) \\ \eta_F(t) \end{pmatrix} + P_B^{-1} \begin{pmatrix} \xi_B(t) \\ \eta_B(t) \end{pmatrix} \right) \quad (6.12)$$

To show that we can always transform the problem so that $B_{21}B_1'=0$, we consider the process

$$\begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} = \begin{pmatrix} B_1 \\ B_{21} \end{pmatrix} u(t) \quad . \quad (6.13)$$

By projecting $w_2(t)$ on $w_1(t)$, one gets

$$\begin{aligned} \tilde{w}_2(t) &= E[w_2(t)|w_1(t)] + \tilde{w}_2(t) \\ &= T_1 w_1(t) + \tilde{w}_2(t) \end{aligned} \quad (6.14)$$

so that

$$\begin{aligned} 0 = E[w_2(t)w_1'(t)] &= E[w_2(t)w_1'(t)] - T_1 E[w_1(t)w_1'(t)] \\ &= B_{21}B_1' - T_1 B_1 B_1' \quad . \end{aligned} \quad (6.15)$$

Then, if we consider the similarity transform

$$\begin{pmatrix} x_1 \\ s_1^* \end{pmatrix} = \underbrace{\begin{pmatrix} I & 0 \\ -T_1 & I \end{pmatrix}}_{T} \begin{pmatrix} x_1 \\ s_1 \end{pmatrix} \quad (6.16)$$
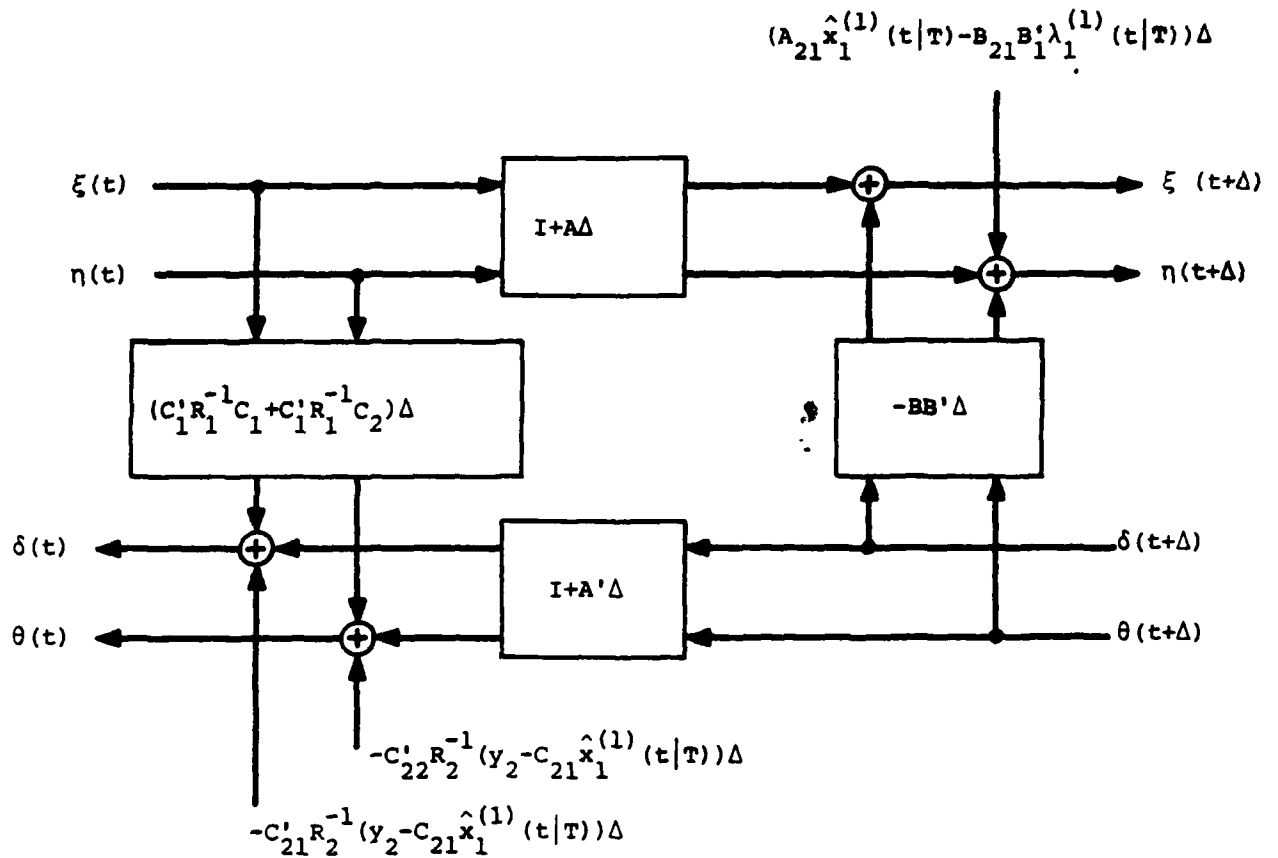
**Figure 6.2.** Scattering layers obtained by superposition.

the system (6.4)-(6.5) becomes

$$
\begin{pmatrix} \dot{x}_1 \\ \dot{s}_1^* \end{pmatrix} = \underbrace{\begin{pmatrix} A_1 & 0 \\ -T_1+A_{21}-T_1A_1+A_{22}T_1 & A_{22} \end{pmatrix}}_{A^*} \begin{pmatrix} x_1 \\ s_1^* \end{pmatrix} + \underbrace{\begin{pmatrix} B_1 \\ B_{21}-T_1B_1 \end{pmatrix}}_{B^*} u
$$

$$
y_1 = \underbrace{\begin{pmatrix} H_1 & 0 \end{pmatrix}}_{C_1^*} \begin{pmatrix} x_1 \\ s_1^* \end{pmatrix} + v_1 \tag{6.17a}
$$

$$
y_2 = \underbrace{(C_{21}+C_{22}T_1C_{22})}_{C_2^*} \begin{pmatrix} x_1 \\ s_1^* \end{pmatrix} + v_2 \tag{6.17b}
$$

where we have $B_{21}^*B_1^{*'} = (B_{21}-T_1B_1)B_1'=0$, so that we can apply the formula (6.9)-6.11) to this system.

Some similar results can be obtained for the <u>filtering estimates</u>. The problem that we shall study is one where the processor $S_1$ can communicate with $S_2$ and where the model available to $S_2$ is the global model, whereas the model available to $S_1$ is of smaller size. In this case, the estimate $\hat{x}_1^{(1)}(t)$ (based on the observations $y_1(\cdot)$) is received by the processor $S_2$ which then wants to combine this estimate with the observations $y_2(\cdot)$ in order to obtain $\hat{x}(t)$. This estimation structure is not symmetric, but in this case the decentralized filter has a simple form which would otherwise be hidden by several changes of basis for the global state-space.

seek below to transform the problem so that $B_{21}B_1'=0$.  When this is
the case, the two-filter solution of the scattering problem is given
by

$$P_s^{-1}(t) \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} = P_F^{-1}(t) \begin{pmatrix} \xi_F(t) \\ \eta_F(t) \end{pmatrix} + P_B^{-1}(t) \begin{pmatrix} \xi_B(t) \\ \eta_B(t) \end{pmatrix} \quad (6.9)$$

where

$$\begin{pmatrix} \dot{\xi}_F(t) \\ \dot{\eta}_F(t) \end{pmatrix} = \left( A - P_F(C_1'R_1^{-1}C_1 + C_2'R_2^{-1}C_2) \right) \begin{pmatrix} \xi_F(t) \\ \eta_F(t) \end{pmatrix} \quad (6.10)$$

$$+ P_F C_2'R_2^{-1}(y_2(t) - C_{21}\hat{x}_1^{(1)}(t|T)) + \begin{pmatrix} 0 \\ A_{21}\hat{x}_1^{(1)}(t|T) \end{pmatrix}$$

$$\begin{pmatrix} \xi_F(0) \\ \eta_F(0) \end{pmatrix} = \begin{pmatrix} 0 \\ \hat{s}_1(0) \end{pmatrix}$$

and

$$-\begin{pmatrix} \dot{\xi}_B(t) \\ \dot{\eta}_B(t) \end{pmatrix} = (-A - P_B(C_1'R_1^{-1}C_1 + C_2'R_2^{-1}C_2 )) \begin{pmatrix} \xi_B(t) \\ \eta_B(t) \end{pmatrix} \quad (6.11)$$

$$+ P_B C_2'R_2^{-1}(y_2(t) - C_{21}\hat{x}_1^{(1)}(t|T)) - \begin{pmatrix} 0 \\ A_{21}\hat{x}_1^{(1)}(t|T) \end{pmatrix}$$

$$P_B^{-1}(T) \begin{pmatrix} \xi_B(T) \\ \eta_B(T) \end{pmatrix}' = 0$$

In the following, we will denote respectively by $P(t)$ and $P_1^{(1)}(t)$

the solutions of the Riccati equations (4.15) and

$$\dot{P}_1^{(1)} = A_1 P_1^{(1)} + P_1^{(1)} A_1' + B_1 B_1' - P_1^{(1)} H_1' R_1^{-1} H_1 P_1^{(1)} , \tag{6.18}$$

where in (4.15) we assume that $A, B, C_1$ and $C_2$ have the block structure

(6.4)-(6.5). The initial conditions for these equations are such that

if

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \tag{6.19}$$

we have $P_{11}(0) = P_1^{(1)}(0)$. Then if we denote by

$$d(t) = \begin{pmatrix} d_1(t) \\ m_1(t) \end{pmatrix} = P^{-1}(t)\hat{x}(t) = P^{-1}(t) \begin{pmatrix} \hat{x}_1 \\ \hat{s}_1(t) \end{pmatrix} \tag{6.20}$$

the information filter form of the centralized filtering estimate, the

global transmission layers associated to $d(t)$ are described in

Figure 6.3a. Similarly, if $d_1^{(1)}(t) = P_1^{(1)-1}(t)\hat{x}_1^{(1)}(t)$, the local

transmission layers associated to $d_1^{(1)}(t)$ are given by Figure 6.3b.

Since $d_1^{(1)}(t)$ is of smaller size than $d(t)$, we cannot apply directly

the principle of superposition to the global and local layers. However,

as for the smoothing problem, we can imbed the local transmission layers

inside the global ones as shown in Figure 6.3c. The vector $(d_1^{(1)'}(t), 0)'$

of transmitted variables has now the same size as $d(t)$. Then, by

and where $P_S$, $P_F$ and $P_B$ are given by (4.12)-(4.14) with the special matrix structure (6.4)-(6.5). Since equations (6.10)-(6.11) depend only on the local estimates $\hat{x}_1^{(1)}(t|T)$ and on the new measurements $y_2(\cdot)$, we can use them to compute the centralized smoothed estimate

$$\hat{x}(t|T) = \begin{pmatrix} \hat{x}_1^{(1)}(t|T) \\ 0 \end{pmatrix} + P_S \left( P_F^{-1} \begin{pmatrix} \xi_F(t) \\ \eta_F(t) \end{pmatrix} + P_B^{-1} \begin{pmatrix} \xi_B(t) \\ \eta_B(t) \end{pmatrix} \right) \quad (6.12)$$

To show that we can always transform the problem so that $B_{21}B_1'=0$, we consider the process

$$\begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} = \begin{pmatrix} B_1 \\ B_{21} \end{pmatrix} u(t) \quad . \quad (6.13)$$

By projecting $w_2(t)$ on $w_1(t)$, one gets

$$\tilde{w}_2(t) = E[w_2(t)|w_1(t)] + \tilde{w}_2(t)$$
$$= T_1 w_1(t) + \tilde{w}_2(t) \quad (6.14)$$

so that

$$0 = E[w_2(t)w_1'(t)] = E[w_2(t)w_1'(t)] - T_1 E[w_1(t)w_1'(t)]$$
$$= B_{21}B_1' - T_1 B_1 B_1' \quad . \quad (6.15)$$

Then, if we consider the similarity transform

$$\begin{pmatrix} x_1 \\ s_1^* \end{pmatrix} = \underbrace{\begin{pmatrix} I & 0 \\ -T_1 & I \end{pmatrix}}_{T} \begin{pmatrix} x_1 \\ s_1 \end{pmatrix} \quad (6.16)$$

(a)



(b)



(c)

**Figure 6.3.** (a) Global transmission layers, (b) Local transmission layers, (c) Imbedding of the local layers.

the system (6.4)-(6.5) becomes

$$
\begin{pmatrix} \dot{x}_1 \\ \dot{s}_1^* \end{pmatrix} = \underbrace{\begin{pmatrix} A_1 & 0 \\ -T_1+A_{21}-T_1A_1+A_{22}T_1 & A_{22} \end{pmatrix}}_{A^*} \begin{pmatrix} x_1 \\ s_1^* \end{pmatrix} + \underbrace{\begin{pmatrix} B_1 \\ B_{21}-T_1B_1 \end{pmatrix}}_{B^*} u
$$

$$
y_1 = \underbrace{\begin{pmatrix} H_1 & 0 \end{pmatrix}}_{C_1^*} \begin{pmatrix} x_1 \\ s_1^* \end{pmatrix} + v_1 \tag{6.17a}
$$

$$
y_2 = \underbrace{(C_{21}+C_{22}T_1C_{22})}_{C_2^*} \begin{pmatrix} x_1 \\ s_1^* \end{pmatrix} + v_2 \tag{6.17b}
$$

where we have $B_{21}^* B_1^{*'} = (B_{21}-T_1B_1)B_1'=0$, so that we can apply the
formula (6.9)-6.11) to this system.

Some similar results can be obtained for the <u>filtering estimates</u>.
The problem that we shall study is one where the processor $S_1$ can com-
municate with $S_2$ and where the model available to $S_2$ is the global model,
whereas the model available to $S_1$ is of smaller size. In this case, the
estimate $\hat{x}_1^{(1)}(t)$ (based on the observations $y_1(\cdot)$) is received by the
processor $S_2$ which then wants to combine this estimate with the
observations $y_2(\cdot)$ in order to obtain $\hat{x}(t)$. This estimation structure is
not symmetric, but in this case the decentralized filter has a simple form
which would otherwise be hidden by several changes of basis for the
global state-space.

superposing the transmission layers of Figures 6.3a and 6.3c, we find in Figure 6.4 that

$$
1(t) = d(t) - \begin{pmatrix} d_1^{(1)}(t) \\ 0 \end{pmatrix} = P^{-1}\hat{x}(t) - \begin{pmatrix} P^{(1)-1}\hat{x}_1^{(1)}(t) \\ 0 \end{pmatrix} \tag{6.21}
$$

satisfies a transmission problem with the same medium as for the global filtering problem and with sources

$$
\tilde{q}(t)\Delta = \left( \begin{pmatrix} P_1^{(1)-1} \\ 0 \end{pmatrix} - P^{-1} \begin{pmatrix} I \\ T_1 \end{pmatrix} \right) B_1 B_1' d_1^{(1)}(t)\Delta + C_2' R_2^{-1} y_2 \Delta \tag{6.22}
$$

(note that $B_{12}B_1' = T_1 B_1 B_1'$). Thus, (t) is given by

$$
\dot{1}(t) = -(A' + P^{-1}BB')1(t) + C_2' R_2^{-1} y_2(t) \tag{6.23}
$$

$$
+ \left( \begin{pmatrix} P_1^{(1)-1} \\ 0 \end{pmatrix} - P^{-1} \begin{pmatrix} I \\ T_1 \end{pmatrix} \right) B_1 B_1' d_1^{(1)}(t)
$$

with initial conditions

$$
1(0) = P^{-1}(0)\hat{x}(0) - \begin{pmatrix} P_1^{(1)-1}(0)\hat{x}_1(0) \\ 0 \end{pmatrix} . \tag{6.24}
$$

If instead of computing $1(t)$ we want to compute

$$
r(t) = P(t)1(t) = \hat{x}(t) - P \begin{pmatrix} P_1^{(1)-1} \\ 0 \end{pmatrix} \hat{x}_1^{(1)}(t) \tag{6.25}
$$

In the following, we will denote respectively by $P(t)$ and $P_1^{(1)}(t)$ the solutions of the Riccati equations (4.15) and

$$\dot{P}_1^{(1)} = A_1 P_1^{(1)} + P_1^{(1)} A_1' + B_1 B_1' - P_1^{(1)} H_1' R_1^{-1} H_1 P_1^{(1)} \, , \qquad (6.18)$$

where in (4.15) we assume that $A, B, C_1$ and $C_2$ have the block structure (6.4)-(6.5). The initial conditions for these equations are such that if

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \qquad (6.19)$$

we have $P_{11}(0) = P_1^{(1)}(0)$. Then if we denote by

$$d(t) = \begin{pmatrix} d_1(t) \\ m_1(t) \end{pmatrix} = P^{-1}(t)\hat{x}(t) = P^{-1}(t) \begin{pmatrix} \hat{x}_1 \\ \hat{s}_1(t) \end{pmatrix} \qquad (6.20)$$

the information filter form of the centralized filtering estimate, the global transmission layers associated to $d(t)$ are described in Figure 6.3a. Similarly, if $d_1^{(1)}(t) = P_1^{(1)-1}(t)\hat{x}_1^{(1)}(t)$, the local transmission layers associated to $d_1^{(1)}(t)$ are given by Figure 6.3b. Since $d_1^{(1)}(t)$ is of smaller size than $d(t)$, we cannot apply directly the principle of superposition to the global and local layers. However, as for the smoothing problem, we can imbed the local transmission layers inside the global ones as shown in Figure 6.3c. The vector $(d_1^{(1)'}(t), 0)'$ of transmitted variables has now the same size as $d(t)$. Then, by

$d_1(t)$

$m_1(t)$

$I - (A'+P^{-1}BB')\Delta$

$d_1(t+\Delta)$

$m_1(t+\Delta)$

$(H_1'R_1^{-1}y_1+C_{21}'R_2^{-1}y_2)\Delta$

$C_{22}'R_2^{-1}y_2\Delta$

(a)

$d_1^{(1)}(t)$

$I - (A_1'+P_1^{(1)-1}B_1B_1')\Delta$

$d_1^{(1)}(t+\Delta)$

$H_1'R_1^{-1}y_1\Delta$

(b)

$d_1^{(1)}(t)$

$I - (A'+P^{-1}BB')\Delta$

$d_1^{(1)}(t+\Delta)$

$0$

$$\left(P^{-1}\begin{pmatrix}B_1B_1'\\B_{12}B_1'\end{pmatrix} - \begin{pmatrix}P_1^{(1)-1}B_1B_1'\\0\end{pmatrix}\right)d_1^{(1)}(t)\Delta + \begin{pmatrix}H_1'R_1^{-1}y_1\Delta\\0\end{pmatrix}$$

(c)

Figure 6.3.    (a) Global transmission layers, (b) Local transmission layers, (c) Imbedding of the local layers.

superposing the transmission layers of Figures 6.3a and 6.3c, we find

in Figure 6.4 that

$$1(t) = d(t) - \begin{pmatrix} d_1^{(1)}(t) \\ \\ 0 \end{pmatrix} = P^{-1}\hat{x}(t) - \begin{pmatrix} P^{(1)-1}\hat{x}_1^{(1)}(t) \\ \\ 0 \end{pmatrix} \qquad (6.21)$$

satisfies a transmission problem with the same medium as for the

without revealing their position. These units tend consequently to

communicate as rarely as possible. In addition, some delays are incurred

when transmitted signals need to be relayed through several communication

nodes before reaching a processing center. Some noise appears also in

## REFERENCES

[1]     J. Zaborsky and J. Singh, "A Reevaluation of the Normal Operating
        State Control of the Power System Using Computer Control and
        System Theory, Part I: Load Estimation," Proc. 1979 IEEE Power
        Industry Computer Applications Conference, pp. 205-213.

[2]     Proceedings of the Third MIT/ONR Workshop on Distributed Information
        and Decision Systems, Vol. IV: Sensors, Surveillance and Tracking
        Issues in $C^3$, Report LIDS-R-1023, Laboratory for Information and
        Decision Systems, MIT, Cambridge, MA., Sept. 1980.

[3]     J.W. Woods and C.H. Radewan, "Kalman Filtering in Two Dimensions,"
        IEEE Trans. Informat. Theory, Vol. IT-23, No. 4, pp. 473-482,
        July 1977.

[4]     Investigation of New Algorithms for Locating and Identifying Spatially
        Distributed Sources and Receivers, Technical Report No. M355-1,
        Information Systems Laboratory, Stanford University, Stanford, CA.,
        March 1979.

[5]     H.T. Kung, "Let's Design Algorithms for VLSI Systems," Proc. Caltech
        Conference on VLSI, pp. 65-90, Jan. 1979.

[6]     S.M. Barta, "On Linear Control of Decentralized Stochastic Systems,"
        Ph.D. Dissertation, Dept. of Elec. Engineering and Comp. Science,
        MIT, Cambridge, MA., July 1978.

[7]     C.W. Sanders, E.C. Tacker, T.D. Linton and R.Y.-S. Ling, "Specific
        Structures for Large-Scale State Estimation Algorithms Having
        Information Exchange," IEEE Trans. Automat. Control, Vol. AC-23,
        No. 2, pp. 255-261, April 1978.

[8]     E.C. Tacker and C.W. Sanders, "Decentralized Structures for State
        Estimation in Large Scale Systems," Large Scale Systems Theory
        and Applications, Vol. 1, No. 1, pp. 39-49, Feb. 1980.

[9]     J.L. Speyer, "Computation and Transmission Requirements for a Decen-
        tralized Linear-Quadratic-Gaussian Control Problem," IEEE Trans.
        Automat. Control, Vol. AC-24, No. 2, pp. 266-269, April 1979.

[10]    C.Y. Chong, "Hierarchical Estimation," Proceedings of the 2rd MIT/ONR
        Workshop on Distributed Communications and Decision Problems, Monterey,
        CA., July 1979, Vol. 1, pp. 205-220.

without revealing their position. These units tend consequently to communicate as rarely as possible. In addition, some delays are incurred when transmitted signals need to be relayed through several communication nodes before reaching a processing center. Some noise appears also in communications when the environment interacts with the transmitted information through distortions, background noise, multipath effects, jamming, etc..

The extension of the results of the previous sections to take into account the existence of delays and of noise in the communication channels is presently under consideration.

We note also that in this paper, no special assumption has been made on the structure of the dynamical systems considered. When there exists a natural separation of the local systems $S_i$ arising either from a weak coupling of these systems, or from a separation between fast and slow time scales, we can use the scattering framework to obtain some decentralized estimation algorithms which exploit this structure, as will be shown in a subsequent paper.

[11] A.S. Willsky, M. Bello, D.A. Castanon, B.C. Levy and G. Verghese, "Combining and Updating of Local Estimates and Regional Maps Along Sets of One-Dimensional Tracks," Report LIDS-R-955, Laboratory for Information and Decision Systems, MIT, Cambridge, MA., Nov. 1979; also to appear in IEEE Trans. Automat. Control.

[12] L. Ljung, T. Kailath and B. Friedlander, "Scattering Theory and Linear Least-Squares Estimation, Part I: Continuous-time Problems," Proc. IEEE, Vol. 64, No. 1, pp. 131-139, Jan. 1976.

[13] B. Friedlander, T. Kailath and L. Ljung, "Scattering Theory and Linear Least-Squares Estimation, Part II: Discrete-time Problems," J. Franklin Inst., Vol. 301, Nos. 1 and 2, pp. 71-82, Jan. 1976.

[14] L. Ljung and T. Kailath, "A Unified Approach to Smoothing Formulas," Automatica, Vol. 12, No. 2, pp. 147-157, March 1976.

[15] G. Verghese, B. Friedlander and T. Kailath, "Scattering Theory and Linear Least-Squares Estimation, Part III: The Estimates," IEEE Trans. Automat. Control, Vol. AC-25, No. 4, pp. 794-802, Aug. 1980.

[16] H.L. Weinert and U.B. Desai, "On Adjoint Models and Fixed-Interval Smoothing," Technical Report 79-8, Elec. Engineering Dept., The Johns Hopkins Univ., Baltimore, MD., 1979.

[17] D.Q. Mayne, "A Solution of the Smoothing Problem for Linear Dynamic Systems," Automatica, Vol. 4, pp. 73-92, 1966.

[18] J.D.C. Fraser, "A New Technique for Optimal Smoothing of Data," Sc.D. Dissertation, MIT, Cambridge, MA., Jan. 1976.

[19] J.E. Wall, A.S. Willsky and N.R. Sandell, "On the Fixed-Interval Smoothing Problem," to appear in Stochastics.

[20] R.M. Redheffer, "Difference Equations in Transmission-Line Theory," in Modern Mathematics for the Engineer, 2nd Series, E.F. Beckenbach, Ed., pp. 282-337, New York: McGraw-Hill, 1961.

[21] R.M. Redheffer, "On the Relation of Transmission-Line Theory to Scattering and Transfer," J. of Mathematics and Physics, Vol. 41, pp. 1-41, 1962.

[22] S.J. Mason and H.J. Zimmerman, Electronic Circuits, Signals and Systems, New York: Wiley, 1960.

[23] W.T. Reid, Riccati Differential Equations, New York: Academic Press, 1972.

[24] G.S. Sidhu and U.B. Desai, "A Method of Orthogonal Directions, Part II: TPBVPs, Scattering Matrices and Estimation Algorithms," Dept. of Elec. Engineering, State Univ. of New York, Buffalo, NY, Tech. Report, 1975.

## REFERENCES

[1]  J. Zaborsky and J. Singh, "A Reevaluation of the Normal Operating
     State Control of the Power System Using Computer Control and
     System Theory, Part I: Load Estimation," Proc. 1979 IEEE Power
     Industry Computer Applications Conference, pp. 205-213.

[2]  Proceedings of the Third MIT/ONR Workshop on Distributed Information
     and Decision Systems, Vol. IV: Sensors, Surveillance and Tracking
     Issues in $C^3$, Report LIDS-R-1023, Laboratory for Information and
     Decision Systems, MIT, Cambridge, MA., Sept. 1980.

[3]  J.W. Woods and C.H. Radewan, "Kalman Filtering in Two Dimensions,"
     IEEE Trans. Informat. Theory, Vol. IT-23, No. 4, pp. 473-482,
     July 1977.

[4]  Investigation of New Algorithms for Locating and Identifying Spatially
     Distributed Sources and Receivers, Technical Report No. M355-1,
     Information Systems Laboratory, Stanford University, Stanford, CA.,
     March 1979.

[5]  H.T. Kung, "Let's Design Algorithms for VLSI Systems," Proc. Caltech
     Conference on VLSI, pp. 65-90, Jan. 1979.

[6]  S.M. Barta, "On Linear Control of Decentralized Stochastic Systems,"
     Ph.D. Dissertation, Dept. of Elec. Engineering and Comp. Science,
     MIT, Cambridge, MA., July 1978.

[7]  C.W. Sanders, E.C. Tacker, T.D. Linton and R.Y.-S. Ling, "Specific
     Structures for Large-Scale State Estimation Algorithms Having
     Information Exchange," IEEE Trans. Automat. Control, Vol. AC-23,
     No. 2, pp. 255-261, April 1978.

[8]  E.C. Tacker and C.W. Sanders, "Decentralized Structures for State
     Estimation in Large Scale Systems," Large Scale Systems Theory
     and Applications, Vol. 1, No. 1, pp. 39-49, Feb. 1980.

[9]  J.L. Speyer, "Computation and Transmission Requirements for a Decen-
     tralized Linear-Quadratic-Gaussian Control Problem," IEEE Trans.
     Automat. Control, Vol. AC-24, No. 2, pp. 266-269, April 1979.

[10] C.Y. Chong, "Hierarchical Estimation," Proceedings of the 2rd MIT/ONR
     Workshop on Distributed Communications and Decision Problems, Monterey,
     CA., July 1979, Vol. 1, pp. 205-220.

[11] A.S. Willsky, M. Bello, D.A. Castanon, B.C. Levy and G. Verghese, "Combining and Updating of Local Estimates and Regional Maps Along Sets of One-Dimensional Tracks," Report LIDS-R-955, Laboratory for Information and Decision Systems, MIT, Cambridge, MA., Nov. 1979; also to appear in IEEE Trans. Automat. Control.

[12] L. Ljung, T. Kailath and B. Friedlander, "Scattering Theory and Linear Least-Squares Estimation, Part I: Continuous-time Problems," Proc. IEEE, Vol. 64, No. 1, pp. 131-139, Jan. 1976.

[13] B. Friedlander, T. Kailath and L. Ljung, "Scattering Theory and Linear Least-Squares Estimation, Part II: Discrete-time Problems," J. Franklin Inst., Vol. 301, Nos. 1 and 2, pp. 71-82, Jan. 1976.

[14] L. Ljung and T. Kailath, "A Unified Approach to Smoothing Formulas," Automatica, Vol. 12, No. 2, pp. 147-157, March 1976.

[15] G. Verghese, B. Friedlander and T. Kailath, "Scattering Theory and Linear Least-Squares Estimation, Part III: The Estimates," IEEE Trans. Automat. Control, Vol. AC-25, No. 4, pp. 794-802, Aug. 1980.

[16] H.L. Weinert and U.B. Desai, "On Adjoint Models and Fixed-Interval Smoothing," Technical Report 79-8, Elec. Engineering Dept., The Johns Hopkins Univ., Baltimore, MD., 1979.

[17] D.Q. Mayne, "A Solution of the Smoothing Problem for Linear Dynamic Systems," Automatica, Vol. 4, pp. 73-92, 1966.

[18] J.D.C. Fraser, "A New Technique for Optimal Smoothing of Data," Sc.D. Dissertation, MIT, Cambridge, MA., Jan. 1976.

[19] J.E. Wall, A.S. Willsky and N.R. Sandell, "On the Fixed-Interval Smoothing Problem," to appear in Stochastics.

[20] R.M. Redheffer, "Difference Equations in Transmission-Line Theory," in Modern Mathematics for the Engineer, 2nd Series, E.F. Beckenbach, Ed., pp. 282-337, New York: McGraw-Hill, 1961.

[21] R.M. Redheffer, "On the Relation of Transmission-Line Theory to Scattering and Transfer," J. of Mathematics and Physics, Vol. 41, pp. 1-41, 1962.

[22] S.J. Mason and H.J. Zimmerman, Electronic Circuits, Signals and Systems, New York: Wiley, 1960.

[23] W.T. Reid, Riccati Differential Equations, New York: Academic Press, 1972.

[24] G.S. Sidhu and U.B. Desai, "A Method of Orthogonal Directions, Part II: TPBVPs, Scattering Matrices and Estimation Algorithms," Dept. of Elec. Engineering, State Univ. of New York, Buffalo, NY, Tech. Report, 1975.